EXISTENTIAL ANALYSIS OF MULTI-POINT BOUNDARY VALUE PROBLEMS VIA GALERKIN'S METHOD

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CERTIFICATE

Certified that the thesis entitled "Existential Analysis of Multi-point Boundary Value Problems Via Galerkin's Method" by M. Venkatesulu has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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CONTENTS

LIST OF SYMBOLS AND NOTATIONS

The following notations are used throughout the work.

sup	supremum
inf	infimum
max	maximum
min	minimum
ess.sup	essential supremum
$\subset$, $\subsetneq$, $\cap$, $\cup$, $\in$	have their usual logical meaning
$\sum_j$	summation with respect to j
J	$[a,b]$, a closed interval on the real line with $b > a$
ϕ	null set
δ_{ij}	Kronecker delta of i and j
$E \times F$	the cartesian product of the sets E and F
D(T), N(T), R(T)	domain, null space, range of the operator T, respectively
$\{\omega_1, \omega_2, \ldots, \omega_m\}$	the set consisting of the elements $\omega_1, \omega_2, \ldots,$ and ω_m
$\langle \omega_1, \omega_2, \ldots, \omega_m \rangle$	the linear space spanned by $\omega_1, \omega_2, \ldots,$ and ω_m
$\{\omega_m\}$	the sequence $(\omega_1, \omega_2, \ldots, \omega_m, \ldots)$
dim E	dimension of the linear space spanned by E
E−F	the set of all points in E which are not members of F
$T: E \to F$	T is a map from the set E into the set F
$T\|E$	the restriction of T to the set E
χ_E	the characteristic function of E

R^n	the n-dimensional real space with Euclidean norm $\|\cdot\|$
$\partial\Omega$, $\bar{\Omega}$	the boundary and the closure of an open bounded subset Ω of R^n, respectively
$\overset{o}{\Omega}$	the maximal open subset of R^n contained in $\bar{\Omega}$ where $\bar{\Omega}$ is an bounded closed subset of R^n
$x^{(n)}$ or $x^{(n)}(\cdot)$	the nth derivative of the real-valued function x or $x(\cdot)$
$x(t+0)$, $x(t-0)$	the right-hand and the left-hand derivative of x at the point t, respectively
$C^n(J)$	the linear space of all n-times continuously differentiable real-valued functions on J
Banach space $C^n(J)$	the linear space $C^n(J)$ equipped with the norm $\|\cdot\|_n$ given by $$\|x\|_n = \max_{i=0,1,\ldots,n} \sup_{t \in J} \|x^{(i)}(t)\|, x \in C^n(J$$
$C^\infty(J)$	the linear space of all infinitely differentiable real-valued functions on J
S	$L_2(J)$, the Hilbert space of all square-integrable real-valued functions on J with the usual inner product and norm denoted through $(\cdot,\cdot)$ and $\|\|\cdot\|\|$, respective
I	the identity operator on S
$E^\perp$	the orthogonal complement of ~~the~~ E in S where E is a subset of S
$E \oplus F$	the direct sum of the subsets E and F of S
$H^n(J)$	$\{x \in C^{n-1}(J) x^{(n-1)}$ is absolutely continuous on J and $x^{(n)} \in S\}$, $n \geq 1$
$H^o(J)$	S
I_n	the identity operator on R^n

Banach space $H^n(J)$	the linear space $H^n(J)$ equipped with the norm $\vert\vert\vert\cdot\vert\vert\vert$ given by $$\vert\vert\vert x\vert\vert\vert = \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t \in J} \vert x^{(i)}(t)\vert\Big) + \vert\vert x^{(n)}\vert\vert,$$ $$x \in H^n(J),\ n \geq 0$$
$H_o^n(J)$	$\{x \in H^n(J)$: x together with its all derivatives upto order $(n-1)$ vanishes at both the end points a and b$\}$, $n \geq 1$
$\tilde{H}^n(J)$	$\{x \in H^n(J)$: $x^{(n)}$ is essentially bounded$\}$,$n \geq 0$ ~~$n \geq 0$~~
μ or $\mu(\cdot)$	the real-valued function on $\tilde{H}^{n-1}(J)$ defined by $$\mu(x) = \max\Big(\max_{i=0,\ldots,n-2}\ \sup_{t \in J} \vert x^{(i)}(t)\vert\ ,$$ $$\operatorname{ess\,sup}_{t \in J} \vert x^{(n-1)}(t)\vert\Big),\ x \in \tilde{H}^{n-1}(J),\ n \geq 1.$$

Synopsis

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'Existential Analysis of Multi-point Boundary Value Problems

Via

Galerkin's Method'.

In recent years the multi-point boundary value problems (MPBVP) are being intensively studied from the point of view of existence theory as well as numerical methods of actually calculating the solutions. There are many areas where these problems arise such as vibrating beam problems with point loadings (W.S. Loud; Pacific J. Math. Vol. 24, No. 2 (1968), 303-317) and difference schemes in numerical analysis (H. Voss; Numer. Math. 24 (1975), 317-329) leading to nonlinear multi-point boundary value problems. Besides, M. Urabe (Numer. Math. 9(1967), 341-366) and H.F. Weinberger (Duke Math. J. 22 (1955), 1-14) gave a wide variety of applications of MPBVP.

A sufficiently general MPBVP for an nth order nonlinear differential equation is of the following form :

$$\tau x \equiv p_n(t) \frac{d^n x}{dt^n} + p_{n-1}(t) \frac{d^{n-1}x}{dt^{n-1}} + \dots + p_o(t)\, x$$

$$= X(t,x,x',\dots,x^{(n-1)}), \qquad (1)$$

$$B_j(x) = \sum_{i=0}^{n-1} (\alpha_{oji}\, x^{(i)}(a) + \alpha_{1ji}\, x^{(i)}(a_1) + \dots + \alpha_{hji} x^{(i)}(b)) = 0 \qquad \dots \quad (2)$$

where $a \leq a_1 \leq a_2 \leq \ldots \leq a_{h-1} \leq b$; $j = 1,2,\ldots,k$.

Earlier, Locker (SIAM J. Appl. Math. 19 (1970), 199-207) applied Galerkin's method for existential analysis of problem (1) - (2) where X did not contain the derivatives of x and the boundary conditions were prescribed only at two points. Later, M. Urabe (Funkcial. Ekvac. 9 (1966), 43-60) considered the problem of existence of an isolated solution of a MPBVP for a system of first order equations when an approximate solution is known.

The present work gives an existential analysis of the MPBVP (1) - (2). An example is worked out, in detail, to illustrate the method. This analysis is further generalised to include totally nonlinear boundary value problems where the boundary conditions are assumed to be nonlinear as well.

The chapter-wise contents are given as follows :

We denote by L, the differential operator generated by the formal differential operator τ and the boundary conditions B_js.

In the first chapter, some properties of L, $L^{-1} = H$ and L^* are established. Assuming that in the domain of L^*, there exists a sequence forming a complete orthonormal set in $L_2[a,b]$, two sequences of projections P_m, Q_m are defined and certain relations involving L, H, P_m, Q_m are established. Besides, H and $H(I-P_m)$ are provided with integral representations.

In the second chapter, the existence of a solution of the MPBVP is established. Towards this end, the original nonlinear operator equation is converted into an equivalent bifurcation equation, making use of Schauder's as well as Banach's fixed point theorems. In the latter case the bifurcation equation is solved.

In the third chapter, a nonlinear third order equation with linear three-point homogeneous boundary conditions is considered in detail utilizing the theory developed in earlier chapters.

In the fourth chapter, a theorem is established concerning the existence of an isolated solution of (1) - (2) when an approximate solution is known.

Finally, the fifth chapter deals with an existential analysis of equation (1) with nonlinear boundary conditions. This is a distinct advancement over earlier results in this direction, since in all the earlier works the boundary conditions were assumed to be linear in order to facilitate the construction of Green's functions.

CHAPTER 0

INTRODUCTION

0.0. GENERAL

The theory of Boundary Value Problems (BVP) represents one of the most important aspects in the study of Differential Equations. Some of the early fundamental contributions to the linear theory belong to G.D. BIRKHOFF, C.E. WILDER et al. A good account of the classical linear theory can be found in [15] by M.A. NAIMARK. However, in recent years due to technological advances and the impact of high speed computers in particular, make it possible to provide some existential and computational theory for certain important classes of problems. One such class of problems is the multi-point boundary value problems (MPBVP). There are many physical and engineering problems where the MPBVP arise. The vibrating beam problems with point loadings [14] and the difference schemes in numerical analysis [19] are some of them. For a list of a variety of other applications one can refer to M. URABE [18] and H.F. WEINBERGER [20]. The present work is a contribution along these lines where we confine ourselves to MPBVP . This is a logical development of the Galerkin's approach adopted by L. CESARI and his students.

0.1. A BRIEF DISCUSSION ON GALERKIN'S METHOD AND COINCIDENCE DEGREE METHOD IN THE SOLUTION OF NONLINEAR BVP

Galerkin's method was widely used earlier in the solution of linear BVP, both for selfadjoint and non-selfadjoint cases [11, 9]. To our knowledge, this was used for nonlinear BVP first by L. CESARI [2,3]. Other contributors include JACK K. HALE [7], H.W. KNOBLOCH [10], M. URABE [16], J. LOCKER [12,13] and R. KANNAN [8]. Later, M. URABE [17] applied Newton's method for studying a system of first order nonlinear differential equations with linear nonhomogeneous multi-point boundary conditions. There he applied Newton's method to obtain an isolated solution assuming the existence of an approximate solution to the given problem. Further, in [18] he presented a method of computing an approximate solution to a MPBVP when an isolated solution is known to exist.

In recent years, J. MAWHIN has developed the theory of coincidence degree with the help of which some nonlinear BVP with nonlinear boundary conditions have been studied. R.E. GAINES and J. MAWHIN [6] proved a Continuation Theorem for nonlinear BVP and applied it for nonlinear BVP with nonlinear two-point boundary conditions.

0.2. COINCIDENCE DEGREE METHOD VERSUS THE METHOD PRESENTED IN THIS WORK

It is worthwhile to compare the projection method used in this work with the results obtained through the application

of coincidence degree theory. Galerkin's method is in essence a projection method. In most of the cases in practice one deals with orthogonal projections. For example, in the case of selfadjoint differential operators, orthogonal projections are defined through eigenfunctions of the operator. Besides, there are some nonselfadjoint problems for which such sequences can be constructed. This is an essential prerequisite for the application of Galerkin's method. We remark that it is not always necessary to have an infinite sequence of orthogonal projections. Lastly, we also remark that in some cases one need not demand orthogonality from the projections. Indeed, what one needs is a sufficiently good approximate solution of the problem so that other methods can be locally applied.

Usually, in the application of degree theory (coincidence degree or otherwise) one should have the existence of an open bounded set on the boundary of which the operator equation does not have a solution. Moreover, one should be able to obtain a suitable homotopic map with nonzero degree with respect to the set and a point in the set. But to our knowledge, there is no suitable procedure for construction of such sets. Here lies the main advantage of the Galerkin's approach. This is precisely the motivation for consistently using Galerkin's approximations in our analysis of MPBVP.

0.3. MOTIVATION FOR THE PRESENT WORK

J. LOCKER [12] developed an existential analysis via Galerkin's method for the equation $Lx = Nx$ where L is an unbounded linear operator defined on a linear manifold of a Hilbert space, and N is a nonlinear operator defined on some subset of the same space. Later, he applied the theory developed in [12] to an n^{th} order nonlinear differential equation with linear homogeneous two-point boundary conditions where the nonlinear part didn't contain any derivative of the unknown variable [13]. This very fact, and that the boundary conditions are prescribed only at two points and are linear homogeneous, provided the motivation for this thesis. In the present work, we develop a theory via Galerkin's method for obtaining the existence of a solution to an n^{th} order nonlinear differential equation with linear homogeneous, as well as nonlinear multi-point boundary conditions where the nonlinear part of the differential equation contains derivatives of the unknown variable. Examples are worked out in detail to illustrate the methods. Moreover, a theorem similar to a theorem of M. URABE [17], concerning the existence of an isolated solution when an approximate solution is known, is given.

0.4. PREREQUISITES

We need the following results from analysis for our work.

THEOREM 0.1. Let τ be a formal differential operator of order n with S coefficients. Then for any sequence $\{f_n\}$ in $H^n(J)$, $\{f_n\}$ and $\{\tau f_n\}$ converge in the topology of S if and only if the sequence $\{f_n\}$ converges in the topology of $H^n(J)$. Moreover, the two norms $|||\cdot|||$ and $||\cdot||_1 = ||\cdot|| + ||\tau\cdot||$ on $H^n(J)$ are equivalent.

For a method of proof of the above theorem one can refer to DUNFORD and SCHWARTZ [5].

THEOREM 0.2. Let the integers m_1 and m_2 be such that $m_1 \geq m_2$. Suppose $\epsilon > 0$ is a real number. Let

$$U = \{\xi \in R^{m_1} : |\xi - \xi_0| \leq \epsilon\} \text{ where } \xi_0 \in R^{m_1}.$$

Further, suppose $\Psi : U \to R^{m_2}$ with $\Psi(\xi_0) = 0$ such that Ψ has first order continuous partial derivatives in the interior of U and the Jacobian matrix of Ψ has rank m_2 at ξ_0.

Then there exists a number $\delta > 0$ and a continuous map Λ such that the set

$$W = \{u \in R^{m_2} : |u| \leq \delta\}$$

is a subset of $\Psi(U)$ and $\Lambda : W \to U$ with $\Psi\Lambda(u) = u$ for all $u \in W$.

For a method of proof one can refer to DIEUDONNE [4].

THEOREM 0.3. Let Ω be an open bounded set in R^m containing the origin. Suppose $f: \bar{\Omega} \to R^m$ is a continuous map.

Then degree of f with respect to Ω and the origin, denoted by $d(f, \Omega, 0)$, has the following properties:

(i) $d(I_m, \Omega, 0) = 1$

(ii) $d(f, \Omega, 0) = 1$ for all f satisfying

$$\sup_{\xi \in \bar{\Omega}} |f(\xi)-\xi| < \inf_{\xi \in \partial\bar{\Omega}} |\xi|$$

(iii) $d(f, \Omega, 0) \neq 0$ implies that $f(\xi) = 0$ has a solution in Ω.

For a proof one can refer to BERGER and BERGER [1].

0.5. MPBVP, AND ASSUMPTIONS USED THROUGHOUT THE WORK

A sufficiently general MPBVP for an n^{th} order nonlinear differential equation is of the following form:

$$\tau x \equiv p_n(t)\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + p_o(t)x$$

$$= X(t,x,x^{(1)},\dots,x^{(n-1)}), \tag{0.1}$$

$$B_j(x) \equiv \sum_{i=0}^{n-1} \left(\alpha_{oji}x^{(i)}(a)+\alpha_{1ji}x^{(i)}(a_1)+\dots+\alpha_{hji}x^{(i)}(b)\right)=0 \tag{0.2}$$

where $a \leq a_1 \leq a_2 \leq \dots \leq a_{h-1} \leq b$; $j=1,2,\dots,k$; $k \leq n$.

A fairly general n^{th} order nonlinear differential equation with nonlinear boundary conditions is of the following form:

$$\tau x \equiv p_n(t)\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + p_o(t)x$$

$$= X(t,x,x^{(1)},\dots,x^{(n-1)}), \tag{0.3}$$

$$f_j(x) \equiv g_j(x(a), x^{(1)}(a), \ldots, x^{(n-1)}(a); x(a_1), x^{(1)}(a_1), \ldots, x^{(n-1)}(a_1); \ldots; x(b), x^{(1)}(b), \ldots, x^{(n-1)}(b)) = 0 \tag{0.4}$$

where $a \le a_1 \le a_2 \le \ldots \le a_{h-1} \le b$; $j=1,2,\ldots,k$; $k \le n$.

We develop an existential analysis for the problems (0.1)-(0.2) and (0.3)-(0.4) with the following basic assumptions:

(i) Each coefficient function $p_i \in C^{\infty}(J)$, $i=0,\ldots,n$ and $p_n(t) \neq 0$ on J.

(ii) The nonlinear function $X(t, x_0, \ldots, x_{n-1})$ is defined for $t \in J$ and $|x_i| \le R_i$, $i=0,\ldots,n-1$ where each $R_i > 0$.

(iii) There exists real number $k_0 \ge 0$ such that for $|x_i| \le R_i$ and $|y_i| \le R_i$, we have

$$|X(t, x_0, x_1, \ldots, x_{n-1}) - X(t, y_0, y_1, \ldots, y_{n-1})| \le k_0 \left(\sum_{i=0}^{n-1} |x_i - y_i| \right), \quad t \in J.$$

(iv) $\alpha_{1ji}, \ldots, \alpha_{hji}$ are real constants such that B_js are linearly independent.

(v) g_js are real-valued functions (not necessarily linear).

(vi) There exist constants $\ell_j \ge 0$ such that for $x, y \in C^{n-1}(J)$, we have

$$|f_j(x) - f_j(y)| \le \ell_j \left(\max_{i=0,\ldots,n-1} \sup_{t \in J} |x^{(i)}(t) - y^{(i)}(t)| \right).$$

0.6. OUTLINE OF THE THESIS

The present thesis is divided into five chapters. The chapter-wise contents are given as follows:

In the first chapter, we deal with the differential operator L generated by the formal differential operator τ and the boundary forms B_js. We define two operators $T_1(\tau)$ and $T_0(\tau)$ generated by τ on some linear manifolds of S. After noticing some properties of L, we show that L has a one-to-one continuous right inverse H. Denoting L^* (which exists) to be the adjoint of L and assuming the existence of a sequence in $D(L^*)$ which forms a complete orthonormal set in S, we define two sequences of projections $\{P_m\}$ and $\{Q_m\}$, and establish certain relations involving L, H, P_m and Q_m. Lastly, we derive certain integral representation for H and $H(I-P_m)$.

In the second chapter, the existence of a solution of the MPBVP (0.1)-(0.2) is established. Towards this end, the original nonlinear operator equation $Lx = Nx$ where $(Nx)(t) = X(t,x(t),\ldots,x^{(n-1)}(t))$ is converted to an equivalent bifurcation equation by making use of Schauder's as well as Banach's fixed point theorems. In the latter case the bifurcation equation is solved.

In the third chapter, we utilize the theory developed in the earlier chapters and prove the existence of a solution

to the following third order nonlinear differential equation with linear homogeneous Three two-point boundary conditions:

$$x^{(3)} = (xx^{(1)})^2 + t - \frac{2}{\pi} \sin \pi t,$$

$$x^{(1)}(0) = x^{(1)}(1) = x(1/2) = 0$$

over the closed interval $[0,1]$.

In the four chapter, a theorem is proved concerning the existence of an isolated solution of (0.1)-(0.2) when an approximate solution is known.

Finally, the fifth chapter deals with an existential analysis of the nonlinear MPBVP (0.3)-(0.4). We make use of the theory developed in this chapter and prove the existence of a solution to the following second order nonlinear differential equation with nonlinear three-point boundary conditions:

$$x^{(2)} + x = \frac{x^3}{2},$$

$$\frac{1}{8}(x^{(1)}(0)-x^{(1)}(\pi))^3 - x^{(1)}(2\pi) = 0,$$

$$\frac{1}{8}(x(0) + x(2\pi))^3 + x(\pi) = 0$$

over the closed interval $[0,2\pi]$.

CHAPTER 1

INTEGRAL REPRESENTATION OF THE OPERATORS 'H' AND '$H(I-P_m)$'

1.0. OUTLINE OF THE CHAPTER

In this chapter we follow the technique of Locker [13] and deal with the differential operator $L:D(L) \subset S \to R(L) \subset S$ generated by a formal differential operator τ and a set of k linearly independent boundary forms B_js. We define two operators $T_1(\tau)$ and $T_o(\tau)$ generated by τ on some subsets of S. After noticing some properties of L, we show that L has a continuous right inverse H. Denoting L^* (which exists) to be the adjoint of L, we assume the existence of a sequence $\{\omega_m\} \subset D(L^*)$ which forms a complete orthonormal set in S such that the set $\{\omega_1, \omega_2, \ldots, \omega_q\}$ forms a basis for $N(L^*)$. We consider the set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ which forms a basis for $N(T_1(\tau))$ such that the set $\{\phi_1, \phi_2, \ldots, \phi_p\}$ forms a basis for $N(L)$. In S, we define two sequences of projections $\{P_m\}$ with $R(P_m) = \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$ and $\{Q_m\}$ with $R(Q_m) = \langle \phi_1, \ldots, \phi_p, H\omega_{q+1}, \ldots, H\omega_m \rangle$ where $m > q$. We also establish certain relations involving L, H, P_m and Q_m. Finally, we present the essential results of the chapter that is the integral representation for H and $H(I-P_m)$.

1.1. DEFINITION AND ELEMENTARY PROPERTIES OF L, $T_1(\tau)$, $T_o(\tau)$ AND H

For a formal differential operator

$$\tau = \sum_{i=0}^{n} p_i(t)\left(\frac{d}{dt}\right)^i, \tag{1.1.1}$$

we assume that $p_i \in C^\infty(J)$, $i=0,1,\ldots,n$ and $p_n(t) \neq 0$ on J.

For each $j \in \{1,2,\ldots,k\}$, we define

$$B_j(x) = \sum_{i=0}^{n-1}\left(\alpha_{oji}x^{(i)}(a)+\alpha_{1ji}x^{(i)}(a_1)+\ldots+\alpha_{hji}x^{(i)}(b)\right) \tag{1.1.2}$$

where $a = a_o \leq a_1 \leq a_2 \leq \ldots \leq a_{h-1} \leq a_h = b$. We assume that $k \leq n$. Further, we assume that $\alpha_{oji}, \alpha_{1ji}, \ldots, \alpha_{hji}$ are real constants such that the boundary forms B_js are linearly independent.

For the formal differential operator τ and the set of k linearly independent boundary forms B_js, the differential operator $L : D(L) \subset S \to R(L) \subset S$ is defined as follows:

$$D(L) = \{x \in H^n(J) : B_j(x) = 0, j=1,2,\ldots,k\}, \qquad Lx = \tau x. \tag{1.1.3}$$

Let the linear operators $T_1(\tau)$ and $T_o(\tau)$ be defined as follows:

$$D(T_1(\tau)) = \{x \in H^n(J)\}, \qquad T_1(\tau)x = \tau x \tag{1.1.4}$$

and

$$D(T_0(\tau)) = \{ x \in H_0^n(J) \} ,$$
$$T_0(\tau)x = \tau x. \tag{1.1.5}$$

Clearly, $T_1(\tau)$ is an extension of both $T_0(\tau)$ and L. Moreover, $T_1(\tau)$ is the adjoint of the operator $T_0(\tau^*)$ where τ^* denotes the formal adjoint of τ. Therefore, $T_1(\tau)$ is a closed linear operator.

Let us recollect the following well known facts about L:

(i) $D(L)$ is dense in S.

(ii) L is a closed linear operator.

(iii) $R(L)$ is closed in S.

(iv) $S = R(L) \oplus N(L^*)$ where L^* denotes the adjoint of L.

(v) $\dim N(L) = p < \infty$ and $\dim N(L^*) = q < \infty$.

Infact, $q \leq p \leq n$ and $n-k = p-q$.

Proofs of (i), (ii), (iii) and (iv) can verified easily and a proof of (v) is given in the appendix 1.

We know that the null space of $T_1(\tau)$ is n-dimensional with $N(L) \subseteq N(T_1(\tau))$. Let us choose functions $\phi_1, \phi_2, \ldots, \phi_n \in C^\infty(J)$ to form an orthonormal basis for $N(T_1(\tau))$ in suchaway that $\phi_1, \phi_2, \ldots, \phi_p$ form an orthonormal basis for $N(L)$. We also choose elements $\omega_1, \omega_2, \ldots, \omega_q \in D(L^*)$ to form an orthonormal basis for $N(L^*)$.

We note that the operator $L|D(L) \cap N(L)^\perp$ is a one-to-one closed linear operators having the same range as L. Let

H denote the inverse of this operator:

$$H = (L|D(L) \cap N(L)^{\perp})^{-1}. \tag{1.1.6}$$

By the Closed Graph Theorem, H is a one-to-one continuous linear operator. Clearly, $D(H) = R(L)$ and $R(H)=D(L) \cap N(L)^{\perp}$. Moreover,

$$LHy = y \quad \text{for all} \quad y \in R(L) \tag{1.1.7}$$

and

$$HLx = x - \sum_{i=1}^{p} (x, \phi_i)\phi_i \quad \text{for all} \quad x \in D(L). \tag{1.1.8}$$

Thus H is a continuous right inverse of L.

1.2. PROJECTIONS P_m, Q_m AND THEIR RELATIONS WITH L AND H

We assume that there exist elements $\omega_{q+1}, \omega_{q+2}, \ldots, \omega_m, \ldots$ belonging to $D(L^*)$ such that the sequence of functions $\omega_1, \omega_2, \ldots, \omega_q, \omega_{q+1}, \omega_{q+2}, \ldots, \omega_m, \ldots$ form a complete orthonormal set in S. Since $S = R(L) \oplus N(L^*)$, the elements ω_{q+1}, $\omega_{q+2}, \ldots, \omega_m, \ldots$ belong to $R(L)$. Hence $H\omega_{q+i}$, $i \geq 1$ are defined and belong to $D(L) \cap N(L)^{\perp}$. Let S_o be the subspace spanned by the elements $\phi_1, \phi_2, \ldots, \phi_p$ and $H\omega_{q+1}, \ldots, H\omega_m$. That is

$$S_o = \langle \phi_1, \phi_2, \ldots, \phi_p, H\omega_{q+1}, \ldots, H\omega_m \rangle . \tag{1.2.1}$$

Since $\phi_i \in N(L)$, $i=1,2,\ldots,p$; $H\omega_{q+i} \in N(L)^{\perp}$, $i \geq 1$ and H is one-to-one, we observe that $\phi_1, \phi_2, \ldots, \phi_p, H\omega_{q+1}$, $H\omega_{q+2}, \ldots, H\omega_m, \ldots$ are linearly independent. Therefore S_o has dimension $p+m-q$.

The sequences of projections $\{P_m\}$ and $\{Q_m\}$ on S are defined as follows:

$$P_m x = \sum_{i=1}^{m} (x, \omega_i)\omega_i \quad \text{for all} \quad x \in S \tag{1.2.2}$$

and

$$Q_m x = \sum_{i=1}^{p} (x, \phi_i)\phi_i + \sum_{i=q+1}^{m} (x, L^* \omega_i) H\omega_i \text{ for all } x \in S \tag{1.2.3}$$

where $m > q$.

The operators P_m and Q_m have the following properties:

(i) P_m and Q_m are continuous linear operators defined on all of S.

(ii) $R(P_m) = \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$.

(iii) $R(Q_m) = S_o \subset D(L)$.

(iv) $P_m^2 = P_m$ and $Q_m^2 = Q_m$.

(v) The range of $(I-P_m)$ is a subset of $R(L)$ and $H(I-P_m)$ is a continuous linear operator defined on all of S.

Proofs of the above statements are trivial.

We now prove an important theorem which we shall make use of in the subsequent discussions.

THEOREM 1.1. The following relations are valid:

(i) $H(I-P_m)Lx = (I-Q_m)x$ for all $x \in D(L)$.

(ii) $LH(I-P_m)x = (I-P_m)x$ for all $x \in S$.

(iii) $LQ_m x = P_m Lx$ for all $x \in D(L)$.

(iv) $Q_m H(L-P_m)x = 0$ for all $x \in S$.

Proof: (i) Let $x \in D(L)$. Then

$$(I-P_m)Lx = Lx - \sum_{i=1}^{m} (Lx,\omega_i)\omega_i$$

$$= Lx - \sum_{i=1}^{m} (x,L^*\omega_i)\omega_i$$

$$= Lx - \sum_{i=q+1}^{m} (x,L^*\omega_i)\omega_i \text{ (since } \omega_1,\ldots,\omega_q \in N(L^*)).$$

Therefore,

$$H(I-P_m)Lx = HLx - \sum_{i=q+1}^{m} (x,L^*\omega_i)H\omega_i$$

$$= x - \sum_{i=1}^{p} (x,\phi_i)\phi_i - \sum_{i=q+1}^{m} (x,L^*\omega_i)H\omega_i \text{ (by(1.1.8))}$$

$$= (I-Q_m)x.$$

(ii) Since $(I-P_m)x \in R(L)$ for all $x \in S$, from (1.1.7) it follows that

$$LH(I-P_m)x = (I-P_m)x.$$

(iii) Let $x \in D(L)$. Then

$$Q_m x = \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=q+1}^{m} (x,L^*\omega_i)H\omega_i .$$

Therefore

$$LQ_m x = \sum_{i=1}^{p} (x,\phi_i)L\phi_i + \sum_{i=q+1}^{m} (x,L^*\omega_i)LH\omega_i$$

$$= \sum_{i=q+1}^{m} (x,L^*\omega_i)\omega_i \text{ (since } \phi_1,\ldots,\phi_p \in N(L) \text{ and } H \text{ is the right inverse of } L).$$

Thus

$$LQ_m x = \sum_{i=1}^{m} (x, L^* \omega_i)\omega_i \quad (\text{since} \quad \omega_1, \ldots, \omega_q \in N(L^*))$$

$$= \sum_{i=1}^{m} (Lx, \omega_i)\omega_i = P_m Lx.$$

(iv) Let $x \in S$. Then

$$H(I-P_m)x = H(x - \sum_{j=1}^{m} (x, \omega_j)\omega_j)$$

$$= Hx - \sum_{j=1}^{m} (x, \omega_j)H\omega_j \ .$$

Therefore,

$$Q_m H(I-P_m)x = Q_m Hx - Q_m(\sum_{j=1}^{m} (x, \omega_j)H\omega_j)$$

$$= \sum_{i=1}^{p} (Hx, \phi_i)\phi_i + \sum_{i=q+1}^{m} (Hx, L^* \omega_i)H\omega_i$$

$$- \sum_{i=1}^{p} (\sum_{j=1}^{m} (x, \omega_j)H\omega_j, \phi_i)\phi_i$$

$$- \sum_{i=q+1}^{m} (\sum_{j=1}^{m} (x, \omega_j)H\omega_j, \ L^* \omega_i)H\omega_i$$

$$= \sum_{i=q+1}^{m} (Hx, \ L^* \omega_i)H\omega_i$$

$$- \sum_{i=q+1}^{m} (\sum_{j=1}^{m} (x, \omega_j) \omega_j, \omega_i)H\omega_i \ .$$

The last step follows from the fact that Hx, and $H\omega_i$, $i=1,2,\ldots,m,\ldots$ belong to $N(L)^{\perp}$, and $\phi_1, \phi_2, \ldots \phi_p \in N(L)$.

Thus

$$Q_m H(I-P_m)x = \sum_{i=q+1}^{m} (LHx, \omega_i) H\omega_i - \sum_{i=q+1}^{m} (x, \omega_i) H\omega_i = 0.$$

This completes the proof of the theorem.

1.3. DERIVATION OF CERTAIN INTEGRAL REPRESENTATION FOR H AND $H(I-P_m)$

Consider the $n \times n$ matrix $\Phi(t)$ which has $\phi_j^{(i-1)}(t)$ as its entry in the i^{th} row and j^{th} column; $i, j = 1, 2, \ldots, n$. For each $t \in J$ this matrix is known to be nonsingular. If we compute Φ^{-1} by forming the adjoint matrix of Φ, then the entry in the j^{th} row and n^{th} column of $\Phi^{-1}(t)$ is just $\frac{W_j(t)}{\det \Phi(t)}$ where $W_j(t)$ is the determinant of the matrix obtained from $\Phi(t)$ by replacing the j^{th} column by $(0, 0, \ldots, 1)$. Thus for each $t \in J$, we have

$$\sum_{j=1}^{n} \frac{\phi_j^{(i)}(t)\, W_j(t)}{\det \Phi(t)} = \begin{cases} 0 & \text{for } i = 0, \ldots, n-2, \\ 1 & \text{for } i = n-1. \end{cases} \qquad (1.3.1)$$

Let $G(\cdot,\cdot)$ be the function defined on the square $J \times J$ by

$$G(t,s) = \sum_{j=1}^{n} \frac{\phi_j(t)\, W_j(s)}{p_n(s) \det \Phi(s)}, \quad a \leq t,\ s \leq b. \qquad (1.3.2)$$

Clearly $G(\cdot,\cdot)$ is a continuous function on $J \times J$ and $G(\cdot, s) \in H^n(J)$.

We need the following lemma.

LEMMA ~~1.1~~1.1. Let $y \in S$, and let

$$u(t) = \int_a^t G(t,s)\, y(s)ds \quad \text{for all } t \in J. \tag{1.3.3}$$

Then the function $u \in H^n(J)$ and $\tau u = y$.

Proof: Since $G(\cdot,s) \in H^n(J)$, clearly $u \in H^n(J)$. On the other hand, differentiating (1.3.3) we get

$$u^{(1)}(t) = \sum_{j=1}^{n} \frac{\phi_j(t)\, W_j(t)y(t)}{p_n(t)\det \Phi(t)} + \int_a^t \sum_{j=1}^{n} \frac{\phi_j^{(1)}(t)W_j(s)}{p_n(s)\det \Phi(s)}\, y(s)ds.$$

But, by relation (1.3.1), the first term on the right hand side of the above equation is zero. Therefore

$$u^{(1)}(t) = \int_a^t \sum_{j=1}^{n} \frac{\phi_j^{(1)}(t)W_j(s)}{p_n(s)\det \Phi(s)}\, y(s)ds.$$

Proceeding similarly, we have

$$u^{(i)}(t) = \int_a^t \sum_{j=1}^{n} \frac{\phi_j^{(i)}(t)W_j(s)}{p_n(s)\det \Phi(s)}\, y(s)ds \text{ for } i=0,1,2,\ldots,n-1.$$

In deriving the above relations we repeatedly make use of (1.3.1). Also, differentiating $u^{(n-1)}$ we get

$$u^{(n)}(t) = \sum_{j=1}^{n} \frac{\phi_j^{(n-1)}(t)W_j(t)y(t)}{p_n(t)\det \Phi(t)} + \int_a^t \sum_{j=1}^{n} \frac{\phi_j^{(n)}(t)W_j(s)}{p_n(s)\det \Phi(s)}\, y(s)ds$$

$$= \frac{y(t)}{p_n(t)} + \int_a^t \sum_{j=1}^{n} \frac{\phi_j^{(n)}(t)W_j(s)}{p_n(s)\det \Phi(s)}\, y(s)ds \quad (\text{by}(1.3.1)).$$

Multiplying $u^{(i)}(t)$ with $p_i(t)$ and summing up with respect to the index 'i' we get

$$(\tau u)(t) = y(t) + \int_a^t \sum_{j=1}^{n} \left[(p_o(t)\phi_j(t)+\dots+p_n(t)\ \phi_j^{(n)}(t)) \frac{W_j(s)\ y(s)}{p_n(s)\det\ \Phi(s)} \right] ds.$$

Therefore, for all $t \in J$ we have

$$(\tau u)(t) = y(t) + \int_a^t \sum_{j=1}^{n} (\tau\phi_j)(t) \frac{W_j(s)\ y(s)}{p_n(s)\det\ \Phi(s)} ds$$

$$= y(t) \qquad (\text{since } \phi_j \in N(T_1(\tau)), j=1,2,\dots,n).$$

This completes the proof of the lemma.

In the next lemma we obtain a matrix $(A_{\ell j})$ which satisfies the equation $(A_{\ell j})\ (B_j(\phi_i)) = \hat{I}$ where $\hat{I}$ is the $(n-p) \times (n-p)$ identity matrix.

LEMMA 1.2. There exist real numbers $A_{\ell j}$, $\ell = p+1,\dots,n$ and $j = 1,2,\dots,k$ such that

$$\sum_{j=1}^{k} A_{\ell j}\ B_j(\phi_i) = \delta_{\ell i} \quad \text{for} \quad \ell, i = p+1,\dots,n.$$

Proof: Let B be the $k \times (n-p)$ matrix with entries $B_j(\phi_i)$ where $j = 1,2,\dots,k$ and $i=p+1,\dots,n$. We assert that B has rank $n-p$. Indeed, let $c_{p+1},\dots,c_n$ be numbers with $\sum_{i=p+1}^{n} B_j(\phi_i)\ c_i = 0$, $j = 1,2,\dots,k$. Take $x(t) = \sum_{i=p+1}^{n} c_i\phi_i(t)$ for $t \in J$. Clearly $x \in D(L)$, $Lx = 0$, and hence $x \in N(L) \cap N(L)^{\perp}$. This implies that $x = 0$ and hence $c_i = 0$, $i = p+1,\dots,n$. Earlier we also noticed that

$k \geq n-p$ (see 1.1(v)). Now, consider any $k \times (k-n+p)$ matrix with linearly independent columns and let the matrix be denoted by D. Let $(B:D)$ be the $k \times k$ matrix formed by the elements of B and D such that the columns of B occupy the first position. Clearly the matrix $(B:D)$ is nonsingular. Hence $(B:D)$ has an inverse. Let the inverse be denoted by A. Thus we have constants $A_{\ell j}$ such that $\sum_{j=1}^{k} A_{\ell j} B_j(\phi_i) = \delta_{\ell i}$ for $\ell, i = p+1, \ldots, n$. This completes the proof of the lemma.

The following theorem gives an integral representation for H.

THEOREM 1.2. Let $y \in R(L)$. Then Hy has a representation given by

$$(Hy)(t) = \sum_{\ell=1}^{n} \phi_\ell(t) \int_a^b \psi_\ell(s) y(s)\,ds + \int_a^t G(t,s)\, y(s)\,ds, \quad t \in J$$

where ψ_ℓ s are defined as follows:

$$\psi_\ell(t) = \begin{cases} -\int_t^b G(s,t)\, \phi_\ell(s)\,ds, \quad \ell = 1,2,\ldots,p, & (1.3.4) \\[2ex] -\sum_{j=1}^{k} \sum_{i=0}^{n-1} \sum_{r=1}^{n} \left[\dfrac{A_{\ell j}\, W_r(t)}{p_n(t) \det \Phi(t)} \cdot (\beta_1\, \chi_{[a,a_1)}(t) + \ldots + \beta_h\, \chi_{(a_{h-1},b]}(t)) \right], & \\ \ell = p+1,\ldots,n. & (1.3.5) \end{cases}$$

Here the constants βs are defined as follows:

$$\beta_1 = \alpha_{1ji}\ \phi_r^{(i)}(a_1)+\dots+\alpha_{hji}\ \phi_r^{(i)}(a_h),$$
$$\beta_2 = \alpha_{2ji}\ \phi_r^{(i)}(a_2)+\dots+\alpha_{hji}\ \phi_r^{(i)}(a_h), \qquad (1.3.6)$$
$$\dots\dots\dots\dots\dots\dots,$$
$$\beta_h = \alpha_{hji}\ \phi_r^{(i)}(a_h).$$

Proof: Take $y \in R(L)$. Let $x = Hy$ and

$$u(t) = \int_a^t G(t,s)y(s)ds,\ t \in J.$$

Using basic properties of H together with lemma ~~1.1.~~ 1.1, we notice that $x,u \in H^n(J)$ and $\tau(x-u) = 0$. Hence $x-u \in N(T_1(\tau))$. Thus

$$x(t) = \sum_{i=1}^{n} c_i\ \phi_i(t) + u(t) \qquad (1.3.7)$$

where c_is are real constants.

Since $x \in N(L)^{\perp}$, we have $(x,\phi_\ell) = 0$ for $\ell=1,2,\dots,p$.

But $(x,\phi_\ell) = (\sum_{i=1}^{n} c_i\,\phi_i,\phi_\ell) + (u,\phi_\ell)$.

Therefore,

$$c_\ell = -(u,\phi_\ell),\quad \ell=1,2,\dots,p.$$

On the other hand,

$$(u,\phi_\ell) = \int_a^b \left(\int_a^t G(t,s)y(s)ds\right)\phi_\ell(t)dt$$

$$= \int_a^b \int_a^t (G(t,s)y(s)\,\phi_\ell(t))ds\ dt$$

$$= \int_a^b \int_s^b G(t,s)y(s)\,\phi_\ell(t)dt\; ds$$

(by Fubini's theorem)

$$= \int_a^b y(s)\left(\int_s^b G(t,s)\;\phi_\ell(t)dt\right)ds$$

$$= -\int_a^b y(s)\;\psi_\ell(s)ds \qquad \text{(by(1.3.4))}$$

Thus

$$c_\ell = -(u,\phi_\ell) = \int_a^b y(s)\psi_\ell(s)ds,\; \ell=1,2,\dots,p. \qquad (1.3.8)$$

Moreover, since $x \in D(L)$, we have $B_j(x) = 0$, $j=1,2,\dots,k$.

But

$$B_j(x) = B_j\left(\sum_{i=1}^{n} c_i\phi_i\right) + B_j(u)$$

$$= \sum_{i=1}^{n} c_iB_j(\phi_i) + B_j(u)$$

$$= \sum_{i=p+1}^{n} c_iB_j(\phi_i) + B_j(u) \quad (\text{since } \phi_1,\phi_2,\dots\phi_p \in N(L)).$$

Thus

$$\sum_{i=p+1}^{n} c_iB_j(\phi_i) + B_j(u) = 0, \quad j = 1,2,\dots,k.$$

Multiplying the above equation with $A_{\ell j}$ and summing with respect to the index 'j' we get

$$\sum_{j=1}^{k}\sum_{i=p+1}^{n} c_iA_{\ell j}B_j(\phi_i) + \sum_{j=1}^{k} A_{\ell j}B_j(u) = 0.$$

That is

$$\sum_{i=p+1}^{n} c_i \sum_{j=1}^{k} A_{\ell j}B_j(\phi_i) + \sum_{j=1}^{k} A_{\ell j}B_j(u) = 0.$$

Hence, by making use of lemma 1.2, we get

$$c_\ell + \sum_{j=1}^{k} A_{\ell j} B_j(u) = 0, \quad \ell = p+1,\ldots,n.$$

Therefore

$$c_\ell = - \sum_{j=1}^{k} A_{\ell j} B_j(u), \qquad \ell = p+1,\ldots,n.$$

But

$$B_j(u) = B_j\left(\int_a^t G(t,s)y(s)ds\right)$$

$$= \sum_{i=0}^{n-1} (\alpha_{oji} u^{(i)}(a) + \ldots + \alpha_{hji} u^{(i)}(b)).$$

Since $u(t) = \int_a^t G(t,s)y(s)ds$, clearly we have

$u(a)=0, u(a_1) = \int_a^{a_1} G(a_1,s)y(s)ds, \ldots,$ and $u(b) = \int_a^b G(b,s)y(s)ds.$

We have already seen in lemma 1.1 that

$$u^{(i)}(t) = \int_a^t \sum_{r=1}^{n} \frac{\phi_r^{(i)}(t)W_r(s)}{p_n(s)\det \Phi(s)} y(s)ds, \quad i=0,\ldots,n-1.$$

Thus

$$u^{(i)}(a) = 0, \quad i=0,\ldots,n-1;$$

$$u^{(i)}(a_1) = \int_a^{a_1} \sum_{r=1}^{n} \frac{\phi_r^{(i)}(a_1)W_r(s)}{p_n(s)\det \Phi(s)} y(s)ds, \qquad i=0,1,\ldots,n-1;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

$$u^{(i)}(a_{h-1}) = \int_a^{a_{h-1}} \sum_{r=1}^{n} \frac{\phi_r^{(i)}(a_{h-1})W_r(s)}{p_n(s)\det \Phi(s)} y(s)ds,$$

$$i=0,1,\ldots,n-1;$$

$$u^{(i)}(b) = \int_a^b \sum_{r=1}^{n} \frac{\phi_r^{(i)}(b)W_r(s)}{p_n(s)\det \Phi(s)}\, y(s)ds, \quad i=0,1,\ldots,n-1.$$

Therefore,

$$\sum_{i=0}^{n-1} (\alpha_{oji}u^{(i)}(a)+\ldots+\alpha_{hji}u^{(i)}(b))$$

$$= \sum_{i=0}^{n-1} (\alpha_{1ji} \int_a^{a_1} \sum_{r=1}^{n} \frac{\phi_r^{(i)}(a_1)W_r(s)}{p_n(s)\det \Phi(s)}\, y(s)ds$$

$$+ \alpha_{2ji} \int_a^{a_2} \sum_{r=1}^{n} \frac{\phi_r^{(i)}(a_2)W_r(s)}{p_n(s)\det \Phi(s)}\, y(s)ds$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \alpha_{hji} \int_a^{b} \sum_{r=1}^{n} \frac{\phi_r^{(i)}(b)W_r(s)}{p_n(s)\det \Phi(s)}\, y(s)ds)$$

$$= \sum_{i=0}^{n-1} \sum_{r=1}^{n} \Big[(\alpha_{1ji}\phi_r^{(i)}(a_1)+\ldots+\alpha_{hji}\phi_r^{(i)}(b)) \int_a^{a_1} \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)}\, ds$$

$$+ (\alpha_{2ji}\phi_r^{(i)}(a_2)+\ldots+\alpha_{hji}\phi_r^{(i)}(b)) \int_{a_1}^{a_2} \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)}\, ds$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \alpha_{hji}\phi_r^{(i)}(b) \int_{a_{h-1}}^{b} \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)} \Big] ds.$$

Thus

$$B_j(u) = \sum_{i=0}^{n-1} \sum_{r=1}^{n} \Big[\beta_1 \int_a^{a_1} \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)}\, ds + \cdots\cdots$$

$$+ \beta_h \int_{a_{h-1}}^{b} \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)}\, ds \Big] \qquad \text{(by (1.3.6))}$$

$$= \sum_{i=0}^{n-1} \sum_{r=1}^{n} \int_a^b \Big[(\beta_1\, \chi_{[a,a_1)}(s) + \cdots + \beta_h\, \chi_{(a_{h-1},b]}(s)) \frac{W_r(s)y(s)}{p_n(s)\det \Phi(s)} \Big] ds.$$

Therefore, from $c_\ell = -\sum_{j=1}^{k} A_{\ell j} B_j(u)$ we get

$$c_\ell = -\sum_{j=1}^{k} \sum_{i=0}^{n-1} \sum_{r=1}^{n} \Big[\int_a^b \frac{A_{\ell j} W_r(s)y(s)}{p_n(s)\det \Phi(s)} (\beta_1\, \chi_{[a,a_1)}(s) + \cdots + \beta_h\, \chi_{(a_{h-1},b]}(s))\, ds \Big],$$

$$\ell = p+1, \ldots, n.$$

Thus

$$c_\ell = \int_a^b y(s)\, \psi_\ell(s), \quad \ell = p+1, \ldots, n \quad \text{(by (1.3.5))}. \qquad (1.3.9)$$

The relations (1.3.7), (1.3.8) and (1.3.9) give the required representation. This completes the proof of the theorem.

Let $K(\cdot,\cdot)$ be the function defined on $J \times J$ by

$$K(t,s) = \begin{cases} \sum_{\ell=1}^{n} \phi_\ell(t)\, \psi_\ell(s) + G(t,s) & \text{for } a \leq s \leq t \leq b, \\ \sum_{\ell=1}^{n} \phi_\ell(t)\, \psi_\ell(s) & \text{for } a \leq t \leq s \leq b. \end{cases} \qquad (1.3.10)$$

We note the following properties of $K(\cdot,\cdot)$:

(i) $K(\cdot,s)$ is continuous on J together with its derivatives upto order $(n-2)$ on J, while the $(n-1)^{th}$ derivative $\frac{\partial^{n-1}K(\cdot,s)}{\partial t^{n-1}}$ is discontinuous at $t=s$ with the jump given by

$$\frac{\partial^{n-1}K(s+o,s)}{\partial t^{n-1}} - \frac{\partial^{n-1}K(s-0,s)}{\partial t^{n-1}} = \frac{1}{p_n(s)} .$$

(ii) For $i=0,1,\ldots,n-1$, the function $\frac{\partial^{i}K(t,\cdot)}{\partial t^{i}}$ is discontinuous at each of the points $s=a_i$. The discontinuities are of first kind and the jumps are continuous functions of t.

We now state a corollary of theorem 1.2.

COROLLARY. The right inverse operator H has an integral representation given by

$$(Hy)(t) = \int_a^b K(t,s)y(s)ds, \; t \in J \tag{1.3.11}$$

for all $y \in R(L)$.

Proof of the above corollary follows immediately from theorem 1.2 and representation (1.3.10).

Let $K_m(\cdot,\cdot)$ be the function defined on $J \times J$ by

$$K_m(t,s)=K(t,s)- \sum_{i=1}^{m} \left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right) \omega_i(s), \; a\leq t,s\leq b. \tag{1.3.12}$$

We notice that $K_m(\cdot,\cdot)$, $\frac{\partial K_m(\cdot,\cdot)}{\partial t}$, ..., $\frac{\partial^n K_m(\cdot,\cdot)}{\partial t^n}$ are square-integrable on $J \times J$, while the functions $\int_a^b \left(\frac{\partial^i K_m(\cdot,s)}{\partial t^i}\right)^2 ds$, $i = 0,1,\ldots,n-1$ are continuous on J.

The following theorem gives an integral representation for the operator $H(I-P_m)$.

THEOREM 1.3. Let $x \in S$. Then

$$(H(I-P_m)x)(t) = \int_a^b K_m(t,s)x(s)ds, \quad t \in J.$$

Proof: Since $(I-P_m)x \in R(L)$, by the corollary of theorem 1.2, we have

$$(H(I-P_m)x)(t) = \int_a^b K(t,s)(I-P_m)x(s)ds. \tag{1.3.13}$$

On the other hand,

$$\begin{aligned}
\int_a^b K_m(t,s)x(s)ds &= \int_a^b \Big[K(t,s) - \sum_{i=1}^m \Big(\int_a^b K(t,\xi)\,\omega_i(\xi)d\xi\Big)\omega_i(s)\Big]x(s)ds \\
&= \int_a^b K(t,s)x(s)ds \\
&\quad - \sum_{i=1}^m \Big(\int_a^b K(t,\xi)\,\omega_i(\xi)d\xi\Big)\int_a^b \omega_i(s)x(s)ds \\
&= \int_a^b K(t,s)x(s)ds \\
&\quad - \sum_{i=1}^m \int_a^b (x,\omega_i)K(t,s)\omega_i(s)ds \\
&= \int_a^b K(t,s)\Big[x(s) - \sum_{i=1}^m (x,\omega_i)\,\omega_i(s)\Big]ds
\end{aligned}$$

$$= \int_a^b K(t,s)(I-P_m)x(s)ds. \tag{1.3.14}$$

Thus, from (1.3.13) and (1.3.14), it follows that

$$(H(I-P_m)x)(t) = \int_a^b K_m(t,s)x(s)ds, \quad t \in J.$$

This completes the proof of the theorem.

CHAPTER 2

AN EXISTENTIAL ANALYSIS FOR A MULTI-POINT BOUNDARY VALUE PROBLEM

2.0. OUTLINE OF THE CHAPTER

In this chapter we follow the notations of chapter 1 and **develop an** existential analysis for the MPBVP $Lx = Nx$ where N is defined subsequently. Indeed, we reduce the equation $Lx = Nx$ to an equivalent bifurcation equation by making use of Schauder's as well as Banach's fixed point theorems. In the latter case we solve the bifurcation equation.

2.1. MPBVP, AND ASSUMPTIONS USED THROUGHOUT THE CHAPTER

We solve the equation $Lx = Nx$ under the following assumptions:

(i) L satisfies all the assumptions of chapter 1.

(ii) Let $X(t,x_o,x_1,\ldots,x_{n-1})$ be a nonlinear real-valued function defined for $t \in J$ and $|x_i| \leq R_i$, $i=0,\ldots,n-1$ where each $R_i > 0$.

(iii) $X(\cdot,x_o,\ldots,x_{n-1}) \in S$ for each fixed $(x_o,\ldots,x_{n-1})$ satisfying $|x_i| \leq R_i$, $i=0,\ldots,n-1$.

(iv) There exists a real number $k_o \geq 0$ such that for $|x_i| \leq R_i$ and $|y_i| \leq R_i$ the function X satisfies the following:

$$|X(t,x_0,\ldots,x_{n-1})-X(t,y_0,\ldots,y_{n-1})| \leq k_0\left(\sum_{i=0}^{n-1} |x_i-y_i|\right), \ t \in J. \tag{2.1.1}$$

We define the operator N as follows:

$$D(N)=\{x \in \tilde{H}^{n-1}(J): \sup_{t \in J}|x^{(i)}(t)| \leq R_i, \ i=0,\ldots,n-2,$$
$$\operatorname{ess.sup}_{t \in J}|x^{(n-1)}(t)| \leq R_{n-1}\}, \tag{2.1.2}$$
$$(Nx)(t) = X(t,x(t),x^{(1)}(t),\ldots,x^{(n-1)}(t)) \text{ for all } t \in J$$
$$\text{for which } |x^{(n-1)}(t)| \leq R_{n-1}.$$

From the assumption (iii) and (2.1.1), it follows easily that $Nx \in S$ for $x \in D(N)$.

We develop an existential theory for the MPBVP

$$Lx = Nx \tag{2.1.3}$$

where N is defined by (2.1.2). In the rest of the chapter we take N to be the operator defined in (2.1.2).

2.2. SOME INEQUALITIES

Let x and $y \in D(N)$. Then

$$||Nx-Ny||=||X(\cdot,x(\cdot),\ldots,x^{(n-1)}(\cdot))-X(\cdot,y(\cdot),\ldots,y^{(n-1)}(\cdot))||$$
$$\leq k_0||\left(\sum_{i=0}^{n-1} |x^{(i)}- y^{(i)}|\right)|| \qquad \text{(by (2.1.1))}$$
$$\leq k_0\left(\sum_{i=0}^{n-1} ||x^{(i)}-y^{(i)}||\right) \quad \text{(by triangle inequality)}$$

$$\leq k_o(\sqrt{b-a}\,(\sum_{i=0}^{n-2}\sup_{t\in J}|x^{(i)}(t)-y^{(i)}(t)|) + ||x^{(n-1)}-y^{(n-1)}||)$$

$$= k_o\ |||x-y|||\ \text{ where } |||\cdot|||\ \text{ is the norm on } H^{n-1}(J).$$

Thus for x and $y \in D(N)$, we have

$$||Nx - Ny|| \leq k_o\ |||x-y|||. \tag{2.2.1}$$

$$\text{Let }\ \tilde{\rho}_m^i = (\sup_{t\in J}\int_a^b (\frac{\partial^i K_m(t,s)}{\partial t^i})^2 ds)^{1/2},\ i=0,\ldots,n-1$$

$$\text{and}\qquad \rho_m^{n-1} = (\int_a^b\int_a^b (\frac{\partial^{n-1}K_m(t,s)}{\partial t^{n-1}})^2 ds\ dt)^{1/2}. \tag{2.2.2}$$

Noting that $(\int_a^b (\frac{\partial^i K_m(t,s)}{\partial t^i})^2 ds)^{1/2}$ is a monotone decreasing sequence for every t, Dini's theorem assures uniform convergence of this sequence to zero as $m \to \infty$ for $i=0,\ldots,n-1$. Thus $\tilde{\rho}_m^i \to 0$ as $m \to \infty$, $i = 0,1,\ldots,n-1$, and $\rho_m^{n-1} \to 0$ as $m \to \infty$.

$$\text{Define }\ \theta_m = \sqrt{b-a}\,(\sum_{i=0}^{n-2}\tilde{\rho}_m^i) + \rho_m^{n-1}$$

$$\text{and}\qquad \bar{\theta}_m = \max_{i=0,\ldots,n-1}\tilde{\rho}_m^i\ . \tag{2.2.3}$$

We observe that both θ_m and $\bar{\theta}_m$ tend to zero as $m \to \infty$.

Let $x \in S$. Then

$$|||\int_a^b K_m(\cdot,s)\,x(s)ds||| = \sqrt{b-a}\,[\sum_{i=0}^{n-2}\sup_{t\in J}|\int_a^b \frac{\partial^i K_m(t,s)}{\partial t^i}x(s)ds|]$$

$$+\left\|\int_a^b \frac{\partial^{n-1}K_m(t,s)}{\partial t^{n-1}}\, x(s)ds\right\|$$

$$\leq \sqrt{b-a}\left[\sum_{i=0}^{n-2}\left(\sup_{t\in J}\int_a^b \left(\frac{\partial^{i}K_m(t,s)}{\partial t^{i}}\right)^2 ds\right)^{1/2}\|x\|\right]$$

$$+\left(\int_a^b\int_a^b \left(\frac{\partial^{n-1}K_m(t,s)}{\partial t^{n-1}}\right)^2 ds\ dt\right)^{1/2}\|x\|$$

(by Schwartz inequality)

$$= \left(\sqrt{b-a}\left[\sum_{i=0}^{n-2}\tilde{\rho}_m^{i}\right]+\rho_m^{n-1}\right)\|x\| \quad \text{(by (2.2.2))}$$

$$= \theta_m\,\|x\| \qquad \text{(by(2.2.3))}.$$

Hence for all $x \in S$ we have

$$\left|\left|\left|\int_a^b K_m(\cdot,s)\, x(s)ds\right|\right|\right| \leq \theta_m\|x\|. \tag{2.2.4}$$

Also

$$\mu\left(\int_a^b K_m(\cdot,s)x(s)ds\right) = \max_{i=0,\ldots,n-1}\ \sup_{t\in J}\left|\int_a^b \frac{\partial^{i}K_m(t,s)}{\partial t^{i}}\, x(s)ds\right|$$

$$\leq \max_{i=0,\ldots,n-1}\left(\sup_{t\in J}\int_a^b \left(\frac{\partial^{i}K_m(t,s)}{\partial t^{i}}\right)^2 ds\right)^{1/2}\|x\|$$

(by Schwartz inequality)

$$= \left(\max_{i=0,\ldots,n-1}\tilde{\rho}_m^{i}\right)\|x\| \quad \text{(by(2.2.2))}$$

$$= \bar{\theta}_m\,\|x\| \qquad \text{(by (2.2.3))}.$$

Therefore for $x \in S$ we have

$$\mu\left(\int_a^b K_m(\cdot,s)\, x(s)ds\right) \leq \bar{\theta}_m\|x\|. \tag{2.2.5}$$

2.3. CONSTRUCTION OF SETS V AND $\tilde{S}_o$

Let us consider the Banach space $H^{n-1}(J)$. We observe that the set $\tilde{H}^{n-1}(J)$ is a linear manifold of $H^{n-1}(J)$. We remember that μ on $\tilde{H}^{n-1}(J)$ is defined as follows:

$$\mu(x) = \max \Big(\max_{i=0,\dots,n-2} \sup_{t\in J} |x^{(i)}(t)|, \operatorname{ess\cdot sup}_{t\in J} |x^{(n-1)}(t)| \Big)$$

for every $x \in \tilde{H}^{n-1}(J)$.

We consider the $p+m-q$-dimensional space S_o. Clearly $S_o \subset D(L) \subset H^n(J)$. We choose $x_o \in S_o$ such that $\beta = \mu(x_o) < R$ where $R = \min_{i=0,\dots,n-1} R_i$. Here R_is are constants in assumption (ii) of section 2.1. Let $z_o = H(I-P_m)Nx_o$, and let e and $\bar{e}$ be real constants such that

$$|||z_o||| \leq e \quad \text{and} \quad \mu(z_o) \leq \bar{e}. \tag{2.3.1}$$

Let c, d, r and $\bar{R}$ be real numbers such that

$$0 < c < d,\ 0 < r < \bar{R},\ c+e < d, \bar{R}+\beta \leq R, \text{ and } r+\bar{e} < \bar{R}. \tag{2.3.2}$$

The sets V and $\tilde{S}_o$ in $\tilde{H}^{n-1}(J)$ are defined as follows:

$$V = \{x \in S_o : |||x-x_o||| \leq c,\ \mu(x-x_o) \leq r\}, \tag{2.3.3}$$

and

$$\tilde{S}_o = \{x \in \tilde{H}^{n-1}(J) : |||x-x_o||| \leq d,\ \mu(x-x_o) \leq \bar{R}\}. \tag{2.3.4}$$

We remember that $|||\cdot|||$ is the norm on $H^{n-1}(J)$. Clearly

$x_0 \in V \subset \tilde{S}_0 \subset D(N)$. Moreover, V and $\tilde{S}_0$ are closed, bounded and convex subsets of $H^{n-1}(J)$.

We observe that V is a closed and bounded subset of S. Indeed, let $\{y_k\}$ be any sequence contained in V and let $\{y_k\}$ converges to y in the topology of S. Firstly, we notice that $\{y_k\} \subset H^n(J)$. Since S_0 is finite-dimensional, the element $y \in S_0$. From the fact that linear operators on a finite-dimensional space are bounded it readily follows that $\{Ly_k\}$ converges to Ly in the topology of S. Hence, by theorem 0.1, the sequence $\{y_k\}$ converges to y in the topology of $H^n(J)$ which implies that the sequence $\{y_k\}$ converges to y in the topology of $H^{n-1}(J)$ and $\mu(y_k - y) \to 0$ as $k \to \infty$. Hence $|||y-x_0||| \leq c$ and $\mu(y-x_0) \leq r$. Thus $y \in V$. Obviously V is a bounded subset of S. Thus V is a closed and bounded subset of S.

2.4. OPERATOR T AND SETS $A(x^*)$ AND A

For each $x^* \in V$, let T be the operator on $\tilde{S}_0$ defined by

$$Tx = x^* + H(I - P_m)\, Nx \tag{2.4.1}$$

for $x \in \tilde{S}_0$. We observe that T is well defined on $\tilde{S}_0$.

For each $x^* \in V$, the set $A(x^*)$ is defined by

$$A(x^*) = \{x \in \tilde{S}_0 : x = Tx\}\,. \tag{2.4.2}$$

We denote by $A = \bigcup_{x^* \in V} A(x^*)$. (2.4.3)

Suppose $A(x^*)$ is non-empty. Then

$$x = Tx = x^* + H(I - P_m)\, Nx \quad \text{for some } x \in \tilde{S}_0.$$

Clearly $x \in D(L)$, and by theorem 1.1 (iv) we have $Q_m\, x = x^*$. Thus

$$Lx = L\, Q_m\, x + L\, H\, (I-P_m)Nx.$$

Using parts (ii) and (iii) of theorem 1.1, we get

$$Lx - Nx = P_m\, (Lx - Nx). \tag{2.4.4}$$

Hence $x \in \tilde{S}_0$ is a solution of (2.1.3), if it satisfies the equation

$$P_m\, (Lx - Nx) = 0. \tag{2.4.5}$$

Equation (2.4.5) is called the bifurcation equation of order m.

We notice that $Lx - Nx = P_m(Lx - Nx)$ on the set A provided A is non-empty. In the following two sections we show that $A(x^*)$ is non-empty. Thus the original MPBVP (2.1.3) will be reduced to equivalent bifurcation equation (2.4.5) on the set A.

2.5. REDUCTION OF THE ORIGINAL MPBVP TO AN EQUIVALENT BIFURCATION EQUATION THROUGH SCHAUDER'S FIXED POINT THEOREM

We need the following lemmas for our discussions in this section.

LEMMA 2.1. Consider the Banach spaces $H^n(J)$ and $H^{n-1}(J)$. Let $\{x_m\}$ be a bounded sequence in the space $H^n(J)$. Then the sequence $\{x_m\}$ has a subsequence which converges in the topology of $H^{n-1}(J)$.

Proof of the above lemma is given in the appendix 2.

REMARK. The injection map from $H^n(J)$ into $H^{n-1}(J)$ is continuous and compact.

LEMMA 2.2. Let the assumptions section 2.1 and conditions (2.3.1) and (2.3.2) be satisfied. Then the operator $T : \tilde{S}_o \to H^n(J)$ is continuous.

Proof : From theorem 1.3, we have

$$Tx = x^* + \int_a^b K_m(\cdot,s) \left[(Nx)(s)\right] ds \text{ for } x \in D(N).$$

Suppose x and $y \in \tilde{S}_o$. Then

$$Tx - Ty = \int_a^b K_m(\cdot,s) \left[(Nx)(s) - (Ny)(s)\right] ds.$$

So,

$$\sqrt{b-a} \left(\sum_{i=0}^{n-1} \sup_{t \in J} |(Tx-Ty)^{(i)}(t)| \right) + ||(Tx-Ty)^{(n)}||$$

$$= \sqrt{b-a} \left(\sum_{i=0}^{n-1} \sup_{t \in J} \left| \int_a^b \frac{\partial^i K_m(t,s)}{\partial t^i} \left[(Nx)(s)-(Ny)(s)\right] ds \right| \right)$$

$$+ \left|\left| \int_a^b \frac{\partial^n K_m(\cdot,s)}{\partial t^n} \left[(Nx)(s)-(Ny)(s)\right] ds \right|\right|$$

$$\leq \sqrt{b-a} \left[\sum_{i=0}^{n-1} \left(\sup_{t \in J} \int_a^b \left(\frac{\partial^i K_m(t,s)}{\partial t^i}\right)^2 ds\right)^{\frac{1}{2}} \right] ||Nx-Ny||$$

$$+ \left(\int_a^b \int_a^b \left(\frac{\partial^n K_m(t,s)}{\partial t^n}\right)^2 ds\, dt\right)^{\frac{1}{2}} ||Nx-Ny||$$

(by Schwartz inequality)

$$= \left[\sqrt{b-a} \left(\sum_{i=0}^{n-1} \rho_m^i \right) + \rho_m^n \right] ||Nx-Ny|| \qquad \text{(by(2.2.2))}$$

where $\rho_m^n = (\int_a^b \int_a^b (\frac{\partial^n K_m(t,s)}{\partial t^n})^2 \, ds \, dt)^{\frac{1}{2}}$.

We clearly notice that $\rho_m^n \to 0$ as $m \to \infty$.

Define $\gamma_m = \sqrt{b-a}\,(\sum_{i=0}^{n-1} \tilde{\rho}_m^i) + \rho_m^n$. Then $\gamma_m \to 0$ as $m \to \infty$.

Therefore, from the above inequality, we get

$$\sqrt{b-a}\,(\sum_{i=0}^{n-1} \sup_{t \in J} |(Tx-Ty)^{(i)}(t)|) + ||(Tx-Ty)^{(n)}||$$

$$\leq \gamma_m \, ||Nx-Ny||$$

$$\leq \gamma_m \, k_o \, |||x-y||| \quad (\text{by}(2.2.1)) . \qquad (2.5.1)$$

Hence the map $T : \tilde{S}_o \to H^n(J)$ is continuous. This completes the proof of the lemma.

COROLLARY. Let the assumptions of section 2.1 and conditions (2.3.1) and (2.3.2) be satisfied. Then the map $T : \tilde{S}_o \to H^{n-1}(J)$ is continuous.

Proof of the corollary follows from (2.5.1) and the fact that

$$|||Tx||| \leq \sqrt{b-a}\,(\sum_{i=0}^{n-1} \sup_{t \in J} |(Tx-Ty)^{(i)}(t)|)+||(Tx-Ty)^{(n)}||.$$

In addition to the assumptions of section 2.1, only for the present case we assume that $|X(t,x_o,x_1,\ldots,x_{n-1})| \leq k_o$ for all $t \in J$, and $|x_i| \leq R_i$, $i = 0,\ldots,n-1$. Thus for $x \in \bar{S}_o$, we have

$$|(Nx)(t)| \leq k_o, \text{ and hence } ||Nx|| \leq k_o \sqrt{b-a} . \qquad (2.5.2)$$

LEMMA 2.3. Suppose the assumptions of section 2.1 and conditions (2.3.1), (2.3.2) and (2.5.2) are satisfied. Then the set $T(\tilde{S}_o)$ is relatively compact in $H^{n-1}(J)$.

Proof : Firstly, we observe that $T(\tilde{S}_o)$ is bounded in $H^n(J)$. Indeed, let $x \in \tilde{S}_o$. Then by theorem 1.3, we have

$$Tx = x^* + \int_a^b K_m(\cdot,s)\,[(Nx)(s)]\,ds.$$

Therefore,

$$\sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |(Tx)^{(i)}(t)|\Big) + ||(Tx)^{(n)}||$$

$$\leq \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |(x^*)^{(i)}(t)|\Big) + ||(x^*)^{(n)}||$$

$$+ \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} \Big|\int_a^b \frac{\partial^i K_m(t,s)}{\partial t^i}\,[(Nx)(s)]\,ds\Big|\Big)$$

$$+ \Big|\Big|\int_a^b \frac{\partial^n K_m(\cdot,s)}{\partial t^n}\,[(Nx)(s)]\,ds\,\Big|\Big|.$$

Then, by a simple calculation as in the proof of lemma 2.2, we get

$$\sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |(Tx)^{(i)}(t)|\Big) + ||(Tx)^{(n)}||$$

$$\leq \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |(x^*)^{(i)}(t)|\Big) + ||(x^*)^{(n)}|| + \gamma_m ||Nx||$$

$$\leq \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |(x^*)^{(i)}(t)|\Big) + ||(x^*)^{(n)}||$$

$$+ \gamma_m k_o \sqrt{b-a} \qquad \text{(by (2.5.2))}.$$

We notice that the right hand side of the above inequality is independent of x. Therefore the set $T(\tilde{S}_o)$ is bounded in $H^n(J)$. Hence, by lemma 2.1, the set $T(\tilde{S}_o)$ is relatively compact in $H^{n-1}(J)$. This completes the proof of the lemma.

We now present a theorem which reduces the original MPBVP to an equivalent bifurcation equation by making use of the Schauder's fixed point theorem.

THEOREM 2.1. Let the assumptions of section 2.1 and conditions (2.3.1), (2.3.2) and (2.5.2) be satisfied. Let 'm' be sufficiently large such that

$$c \leq d - \theta_m k_o \sqrt{b-a} \text{ and } r \leq \bar{R} - \bar{\theta}_m k_o \sqrt{b-a} . \tag{2.5.3}$$

Then for each $x^* \in V$ the set $A(x^*)$ is non-empty. Moreover, $Lx - Nx = P_m(Lx - Nx)$ on the set A.

Proof : By (2.4.2), it is enough to show that the map T corresponding to $x^* \in V$ has a fixed point in $\tilde{S}_o$. We have seen in the corollary of lemma 2.2 and in lemma 2.3 that the map $T : \tilde{S}_o \to H^{n-1}(J)$ is continuous and the set $T(\tilde{S}_o)$ is relatively compact in $H^{n-1}(J)$. We now prove that $T(\tilde{S}_o) \subset \tilde{S}_o$. Let $x \in \tilde{S}_o$. Then $Tx \in H^n(J)$ and

$$Tx = x^* + \int_a^b K_m(\cdot, s) \left[(Nx)(s) \right] ds.$$

Therefore

$$Tx - x_o = x^* - x_o + \int_a^b K_m(\cdot, s) \left[(Nx)(s) \right] ds.$$

Hence

$$|||Tx-x_0||| \le |||x^*-x_0||| + |||\int_a^b K_m(\cdot,s)\,[(Nx)(s)]\,ds|||$$

$$\le c + \theta_m \,||Nx|| \qquad \text{(by (2.3.3) and (2.2.4))}$$

$$\le c + \theta_m k_0 \sqrt{b-a} \qquad \text{(by (2.5.2))}$$

$$\le d \qquad \text{(by (2.5.3))}.$$

Also

$$\mu(Tx-x_0) \le \mu(x^* - x_0) + \mu(\int_a^b K_m(\cdot,s)\,[(Nx)(s)]\,ds)$$

$$\le r + \bar{\theta}_m \,||Nx|| \qquad \text{(by (2.3.3) and (2.2.5))}$$

$$\le r + \bar{\theta}_m k_0 \sqrt{b-a} \qquad \text{(by (2.5.2))}$$

$$\le \bar{R} \qquad \text{(by (2.5.3))}.$$

Thus $T(\tilde{S}_0) \subset \tilde{S}_0$. We also observed that $\tilde{S}_0$ is a closed, bounded and convex subset of $H^{n-1}(J)$. Hence the application of Schauder's fixed point theorem to the pair $\tilde{S}_0$ and T yields that the map T has a fixed point in $\tilde{S}_0$. Thus for each $x^* \in V$ the set $A(x^*)$ is non-empty. Moreover, since $A(x^*)$ is non-empty, we have $Lx - Nx = P_m(Lx-Nx)$ on the set A. Thus we have the desired reduction on the non-empty set A and the proof is completed.

2.6. REDUCTION OF THE ORIGINAL MPBVP TO AN EQUIVALENT BIFURCATION EQUATION THROUGH BANACH'S FIXED POINT THEOREM

We now prove an important theorem which reduces the original MPBVP to an equivalent bifurcation equation by making use of the Banach's fixed point theorem.

THEOREM 2.2. Let the assumptions of section 2.1 and conditions (2.3.1) and (2.3.2) be valid. Let 'm' be sufficiently large such that

$$\theta_m k_o < 1,\ c+e \leq (1-\theta_m k_o)d \text{ and } r+\bar{e} \leq \bar{R} - \bar{\theta}_m k_o d. \quad (2.6.1)$$

Then for each $x^* \in V$ the set $A(x^*)$ is singleton. Moreover, $Lx - Nx = P_m(Lx-Nx)$ on the set A.

Proof : By (2.4.2), it is enough to show that the map T corresponding to $x^* \in V$ has a unique fixed point in $\tilde{S}_o$. Firstly, we show that the set $T(\tilde{S}_o) \subset \tilde{S}_o$. Let $x \in \tilde{S}_o$. Then $Tx \in H^n(J)$ and

$$Tx = x^* + \int_a^b K_m(\cdot,s)\ [(Nx)(s)]\ ds.$$

Therefore

$$Tx - x_o = x^* - x_o + \int_a^b K_m(\cdot,s)\ [(Nx)(s)]\ ds.$$

Hence,

$$|||Tx-x_o||| \leq |||x^*-x_o|||+ |||\int_a^b K_m(\cdot,s)\ [(Nx)(s) - (Nx_o)(s)]\ ds||| + |||z_o|||$$

$$\leq c+\theta_m||Nx-Nx_o||+e \quad \text{(by (2.3.3),(2.2.4) and (2.3.1))}$$

$$\leq c+\theta_m k_o|||x-x_o|||+e \quad \text{(by (2.2.1))}$$

$$\leq c+e + \theta_m k_o d \quad \text{(by (2.3.4))}$$

$$\leq d \quad \text{(by (2.6.1))}.$$

Also

$$\mu(Tx-x_o) \leq \mu(x^*-x_o) + \mu(\int_a^b K_m(\cdot,s) [(Nx)(s)-(Nx_o)(s)] ds) + \mu(z_o)$$

$$\leq r+\bar{\theta}_m||Nx-Nx_o||+\bar{e} \qquad \text{(by (2.3.3),(2.2.5) and (2.3.1))}$$

$$\leq r+\bar{\theta}_m k_o|||x-x_o|||+\bar{e} \qquad \text{(by (2.2.1))}$$

$$\leq r+\bar{e} + \bar{\theta}_m k_o d \qquad \text{(by (2.3.4))}$$

$$\leq \bar{R} \qquad \text{(by (2.6.1))}.$$

Thus $T(\tilde{S}_o) \subset \tilde{S}_o$. We also know that $\tilde{S}_o$ is a closed subset of $H^{n-1}(J)$. We next show that T is a contraction on $\tilde{S}_o$. Let x and y $\in \tilde{S}_o$. Then

$$Tx - Ty = \int_a^b K_m(\cdot,s) [(Nx)(s) - (Ny)(s)] ds.$$

Therefore

$$|||Tx-Ty||| = |||\int_a^b K_m(\cdot,s) [(Nx)(s) - (Ny)(s)] ds|||$$

$$\leq \theta_m ||Nx-Ny|| \qquad \text{(by (2.2.4))}$$

$$\leq \theta_m k_o |||x-y||| \qquad \text{(by (2.2.1))}.$$

Since $\theta_m k_o < 1$, from the above inequality it is evident that the map T is a contraction on $\tilde{S}_o$. Hence the application of Banach's fixed point theorem to the pair $\tilde{S}_o$ and T yields that the map T has unique fixed point in the set $\tilde{S}_o$. Thus for each $x^* \in V$ the set $A(x^*)$ is singleton. Moreover, since $A(x^*)$ is non-empty, we have $Lx-Nx = P_m(Lx-Nx)$ on the set A. Thus we have the desired reduction on the set A and the proof is completed.

2.7. SOLUTION OF THE BIFURCATION EQUATION

Throughout this section the conditions of theorem 2.2 are assumed to be valid. By theorem 2.2 we know that $Lx-Nx = P_m(Lx-Nx)$ on the non-empty set A. In this section we shall find an element $\hat{x} \in A$ such that $P_m(L\hat{x} - N\hat{x}) = 0$. Then obviously $\hat{x}$ is a solution of the original MPBVP (2.1.3). Moreover, $|||\hat{x} - x_o||| \leq d$ and $\mu(\hat{x} - x_o) \leq \bar{R}$.

By theorem 2.2 we know that for each $x^* \in V$ the set $A(x^*)$ is singleton. Let $A(x^*) = \hat{x}$. That is for each $x^* \in V$ there exists a unique element $\hat{x} \in A \subset \tilde{S}_o$ such that

$$\hat{x} = T\hat{x} = x^* + H(I - P_m)\, N\hat{x}\ .$$

We now assert that $\hat{x}$ vary continuously with x^*. Indeed, let x^* and $y^* \in V$, and let $\hat{x}$ and $\hat{y} \in \tilde{S}_o$ be the unique elements with

$$\hat{x} = T\hat{x} = x^* + H(I-P_m)N\hat{x} \text{ and } \hat{y} = T\hat{y} = y^* + H(I-P_m)N\hat{y}.$$

Then

$$\hat{x}-\hat{y} = x^* - y^* + \int_a^b K_m(\cdot,s)\ [(N\hat{x})(s) - (N\hat{y})(s)]\ ds$$

(see theorem 1.3).

Therefore,

$$|||\hat{x}-\hat{y}||| \leq |||x^*-y^*||| + |||\int_a^b K_m(\cdot,s)\ [(N\hat{x})(s)-(N\hat{y})(s)]\, ds|||$$

$$\leq |||x^*-y^*||| + \theta_m\ ||N\hat{x}-N\hat{y}|| \quad \text{(by (2.2.4))}$$

$$\leq |||x^*-y^*||| + \theta_m\ k_o\ |||\hat{x}-\hat{y}||| \text{(by (2.2.1))}.$$

Hence

$$|||\hat{x}-\hat{y}||| \leq (1-\theta_m k_o)^{-1}|||x^*-y^*||| \quad (\text{since } \theta_m k_o < 1 \text{ by } (2.6.1)).$$

Therefore $\hat{x}$ vary with x^* continuously in the space $H^{n-1}(J)$.

Under the hypothesis of theorem 2.2, let $\Gamma : V \to D(L) \cap \tilde{S}_o$ be the continuous operator defined by $\Gamma(x^*) = \hat{x}$ for $x^* \in V$ where $\hat{x}$ is the unique element in $\tilde{S}_o$ which is a fixed point of the operator T corresponding to x^*. We note that $P_m(L\Gamma x^* - N\Gamma x^*)$ is an operator mapping V into the subspace $\langle \omega_1, \omega_2, \ldots, \omega_m \rangle$ of the space S.

We now prove the following lemma.

LEMMA 2.4. Suppose the assumptions of theorem 2.2 are satisfied. Let $\{x_n^*\}$ be any sequence contained in the set V. Suppose $\{x_n^*\}$ converges to x^* in the topology of S. Then the sequence $\{\Gamma x_n^*\}$ converges to Γx^* in the topology of $H^{n-1}(J)$.

Proof : Since $\{x_n^*\}$ converges to x^* in the topology of S, we saw earlier that the sequence $\{x_n^*\}$ also converges to x^* in the topology of $H^n(J)$. Hence $\{x_n^*\}$ converges to x^* in both the norms $|||\cdot|||$ and $\mu(\cdot)$. Moreover, since $\Gamma : V \subset H^{n-1}(J) \to A \subset D(L) \cap \tilde{S}_o \subset H^{n-1}(J)$ is continuous, the sequence $\{\Gamma x_n^*\}$ converges to Γx^* in the topology of $H^{n-1}(J)$. This completes the proof of the lemma.

The next theorem is an immediate corollary of theorem 2.2.

THEOREM 2.3. Let the assumptions of section 2.1 and conditions (2.3.1) and (2.3.2) be valid. Suppose the relations (2.6.1) are

satisfied. If there exists an $x^* \in V$ such that

$$P_m(L\Gamma x^* - N\Gamma x^*) = 0, \qquad (2.7.1)$$

then the element $\hat{x} = \Gamma x^*$ is a solution of the MPBVP $Lx = Nx$. Further, $Q_m \hat{x} = x^*$, $|||\hat{x}-x_o||| \leq d$ and $\mu(\hat{x}-x_o) \leq \bar{R}$.

In theorem 2.3 the problem of solving the equation (2.1.3) has been reduced to the problem of solving the equation (2.7.1). Equation (2.7.1) is generally called the determining equation. Below, we solve the equation (2.7.1) under certain additional assumptions.

Let $\psi : D(L) \cap \tilde{S}_o \to \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$ be the operator defined by

$$\psi x = P_m(Lx - Nx) \qquad (2.7.2)$$

for all $x \in D(L) \cap \tilde{S}_o$.

We note that $V \subset D(L) \cap \tilde{S}_o$. Let x^* and $y^* \in V$. Then

$$\psi\Gamma x^* - \psi\Gamma y^* = P_m(L\Gamma x^* - L\Gamma y^* - N\Gamma x^* + N\Gamma y^*)$$

$$= \sum_{i=1}^{m} (L(\Gamma x^* - \Gamma y^*), \omega_i)\omega_i + P_m(N\Gamma y^* - N\Gamma x^*).$$

Therefore

$$||\psi\Gamma x^* - \psi\Gamma y^*|| \leq ||\sum_{i=1}^{m} (\Gamma x^* - \Gamma y^*, L^*\omega_i)\omega_i|| + ||N\Gamma y^* - N\Gamma x^*||$$

(by Bessel's inequality)

$$\leq \sum_{i=1}^{m} ||\Gamma x^* - \Gamma y^*||\ ||L^*\omega_i|| + k_o|||\Gamma x^* - \Gamma y^*|||$$

(by using Schwartz inequality and relation (2.2.1))

$$\leq (\sum_{i=1}^{m} ||L^*\omega_i|| + k_o)|||\Gamma x^* - \Gamma y^*|||$$

(since $||\cdot|| \leq |||\cdot|||$).

But, in lemma 2.4, we saw that if x^* converges to y^* in the topology of S, then Γx^* converges to Γy^* in the topology of $H^{n-1}(J)$. Thus

$$\psi\Gamma : V \subset S \to \langle \omega_1, \ldots, \omega_m \rangle \subset S$$

is continuous.

Similarly we can show that

$$\psi : V \subset S \to \langle \omega_1, \ldots, \omega_m \rangle \subset S$$

is continuous.

Also, by (2.7.2), the equation (2.7.1) can be rewritten as

$$\psi\Gamma x^* = 0. \tag{2.7.3}$$

Since Γ is defined implicitly, the existence of a solution of $\psi\Gamma x^* = 0$ is better studied through the operator ψ restricted to V.

The following lemma relates these two operators.

<u>LEMMA</u> 2.5. Let the assumptions of section 2.1 and conditions (2.3.1) and (2.3.2) be satisfied. Suppose the relations (2.6.1) are valid. Then for each $x^* \in V$ we have

$$||\psi\Gamma x^* - \psi x^*|| \leq (\theta_m k_o d + e) k_o. \tag{2.7.4}$$

<u>Proof</u> : Suppose $x^* \in V$. Let $x = \Gamma x^*$. Then $x \in D(L) \cap \tilde{S}_o$, $Q_m x = x^*$ and $P_m Lx = P_m Lx^*$. So

$$\psi\Gamma x^* - \psi x^* = P_m(Nx^* - N\Gamma x^*).$$

Hence, by Bessel's inequality and relation (2.2.1), we get

$$
\begin{aligned}
||\psi\Gamma x^* - \psi x^*|| &\le ||Nx^* - N\Gamma x^*|| \\
&\le k_o\, |||x^* - \Gamma x^*||| \\
&= k_o|||H(I-P_m)[N\Gamma x^* - Nx_o] + z_o||| \quad \text{(see (2.4.1))} \\
&\le k_o(|||\int_a^b K_m(\cdot,s)[(N\Gamma x^*)(s)-(Nx_o)(s)ds]||| \\
&\qquad + |||z_o|||) \\
&\le k_o(\theta_m k_o d+e) \quad \text{(see the proof of theorem 2.2).}
\end{aligned}
$$

This completes the proof of the lemma.

We use the above lemma to determine conditions on $\psi|V$ which guarantee that the equation (2.7.1) is solvable. Since ψ and $\psi\Gamma$ both restricted to V map a finite-dimensional space into another finite-dimensional space, we shall define a map which takes one coefficient space into the other.

For the above said purpose, we apply the Gram-Schmidt process to the elements $H\omega_{q+1},\ldots,H\omega_m$ to obtain orthonormal elements $\eta_{q+1},\ldots,\eta_m$. Let $\bar{m} = p + m - q$. By 1.1(v) we have $\bar{m} \ge m$. Let $E^{\bar{m}}$ be a copy of Euclidean $\bar{m}$-space where we represent each point $\xi \in E^{\bar{m}}$ as an $\bar{m}$-tuple : $\xi = (b_1,b_2,\ldots,b_p,\ c_{q+1},\ldots,c_m)$. Also, let E^m be a copy of Euclidean m-space where we represent each $u \in E^m$ as an m-tuple : $u = (u_1,\ldots,u_m)$. We define two operators

$\Gamma_1 : E^{\bar{m}} \to S_o$ and $\Gamma_2 : \langle\omega_1,\ldots,\omega_m\rangle \to E^m$ by

$$\Gamma_1(b_1,b_2,\ldots,b_p,c_{q+1},\ldots,c_m) = \sum_{i=1}^{p} b_i\,\phi_i + \sum_{i=q+1}^{m} c_i\,\eta_i \tag{2.7.5}$$

and

$$\Gamma_2 \left(\sum_{i=1}^{m} u_i \; \omega_i \right) = (u_1, \ldots, u_m). \tag{2.7.6}$$

Clearly Γ_1 and Γ_2 are isomorphisms. Let $\xi_o \in E^{\bar{m}}$ be the element with $\Gamma_1(\xi_o) = x_o$, and let $\Psi : E^{\bar{m}} \to E^m$ be the operator defined by

$$\Psi = \Gamma_2 \; \psi \; \Gamma_1. \tag{2.7.7}$$

Let us choose a number $\varepsilon > 0$ such that the set

$$U = \{\xi \in E^{\bar{m}} : |\xi - \xi_o| \leq \varepsilon\} \tag{2.7.8}$$

is mapped by Γ_1 into the set V. The proof of existence of such an ε is given in the appendix 3. Under the hypothesis of theorem 2.2, we observe that $\Gamma_2 \; \psi \; \Gamma_1$ and $\Gamma_2 \; \psi \; \Gamma\Gamma_1$ map the ball $U \subset E^{\bar{m}}$ continuously into E^m. This is used in the following theorems to establish the existence of a solution to the equation (2.7.3).

THEOREM 2.4. Suppose m = 1. Let the assumptions of section 2.1 and conditions (2.3.1) and (2.3.2) be satisfied. Suppose the conditions (2.6.1) are valid. If there exists a number $\delta > 0$ such that the closed interval $[-\delta, \delta] \subset \Psi(U)$ and $(\theta_m \; k_o \; d + e)k_o \leq \delta$, then there exists $x^* \in V$ satisfying the equation $\psi \Gamma x^* = 0$. Moreover, $\hat{x} = \Gamma x^*$ is a solution of original MPBVP $Lx = Nx$ and $Q_m \; \hat{x} = x^*$. Also $|||\hat{x} - x_o||| \leq d$ and $\mu(\hat{x} - x_o) \leq \bar{R}$.

Proof : Let us choose ξ_1, $\xi_2 \in U$ such that $\Psi(\xi_1) = \delta$ and $\Psi(\xi_2) = -\delta$. Let $x^* = \Gamma_1(\xi_1)$ and $y^* = \Gamma_1(\xi_2)$. Clearly x^* and y^*

are elements of V. By lemma 2.5 we have

$$||\psi\Gamma x^* - \psi x^*|| \leq (\theta_m k_o d + e) k_o \leq \delta \quad \text{and}$$

$$||\psi\Gamma y^* - \psi y^*|| \leq (\theta_m k_o d + e) k_o \leq \delta.$$

Thus

$$\begin{aligned} |\Gamma_2 \psi \Gamma\Gamma_1(\xi_1) - \delta| &= |\Gamma_2 \psi \Gamma\Gamma_1(\xi_1) - \Psi(\xi_1)| \\ &= |\Gamma_2 \psi \Gamma\Gamma_1(\xi_1) - \Gamma_2 \psi \Gamma_1(\xi_1)| \\ &\qquad\qquad \text{(by (2.7.7))} \\ &= ||\psi \Gamma\Gamma_1(\xi_1) - \psi \Gamma_1(\xi_1)|| \\ &= ||\psi\Gamma x^* - \psi x^*|| \\ &\leq \delta. \end{aligned}$$

Therefore $\Gamma_2 \psi \Gamma\Gamma_1(\xi_1) \geq 0$. Similarly we can show that $\Gamma_2 \psi \Gamma\Gamma_1(\xi_2) \leq 0$. Moreover, since $\Gamma_2 \psi \Gamma\Gamma_1$ is a continuous map, the set $\Gamma_2 \psi \Gamma\Gamma_1(U)$ is connected. Hence there exists $\xi \in U$ such that $\Gamma_2 \psi \Gamma\Gamma_1(\xi) = 0$. Then by setting $x^* = \Gamma_1(\xi)$, we have $x^* \in V$ and $\psi\Gamma x^* = 0$. Hence $\hat{x} = \Gamma x^*$ is a solution of $Lx = Nx$. Obviously, as we noted earlier, we have $Q_m \hat{x} = x^*, |||\hat{x}-x_o||| \leq d$ and $\mu(\hat{x}-x_o) \leq \bar{R}$. This completes the proof of the theorem.

We need the following lemma for our next theorem.

<u>LEMMA</u> 2.6. Suppose $\Psi(\xi_o) = 0$ and the following conditions are satisfied :

(i) The map Ψ has first order continuous partial derivatives in the interior of U.

(ii) The Jacobian matrix for Ψ has rank m at ξ_o.

Then there exists a number $\delta > 0$ and a continuous map Λ such that the set

$$\bar{\Omega} = \{u \in E^m : |u| \leq \delta\} \tag{2.7.9}$$

is a subset of $\Psi(U)$ and $\Lambda : \bar{\Omega} \to U$ with $\Psi\Lambda(u) = u$ for all $u \in \bar{\Omega}$.

For a proof of the lemma see the reference given for the proof of theorem 0.2.

We now present a theorem which relaxes the condition $m = 1$.

THEOREM 2.5. Let the assumptions of sections 2.1 and conditions (2.3.1) and (2.3.2) be valid. Let the conditions (2.6.1) be satisfied. Suppose the assumptions of lemma 2.6 are valid and suppose $(\theta_m k_o d + e) k_o < \delta$ where δ is the number in (2.7.9). Then there exists an $x^* \in V$ such that $\psi\Gamma x^* = 0$. Moreover, $\hat{x} = \Gamma x^*$ is a solution of the original MPBVP $Lx - Nx$ and $Q_m \hat{x} = x^*$. Also $|||\hat{x} - x_o||| \leq d$ and $\mu(\hat{x} - x_o) \leq \bar{R}$.

Proof : Let us consider the two maps $\Gamma_2 \psi \Gamma\Gamma_1 \Lambda : \bar{\Omega} \to E^m$ and $I_m : \bar{\Omega} \to E^m$. Take $u \in \bar{\Omega}$ and let $x^* = \Gamma_1 \Lambda(u)$. Then $x^* \in V$ and $\Gamma_2 \psi x^* = \Gamma_2 \psi \Gamma_1 \Lambda(u) = \Psi\Lambda(u) = u$. Moreover,

$$\begin{aligned} |\Gamma_2\psi \Gamma\Gamma_1 \Lambda(u) - u| &= |\Gamma_2 \psi \Gamma\Gamma_1 \Lambda(u) - \Gamma_2 \psi x^*| \\ &= ||\psi\Gamma x^* - \psi x^*|| \\ &\leq (\theta_m k_o d + e)k_o \quad \text{(by lemma 2.5)} \\ &< \delta. \end{aligned}$$

Thus

$$|\Gamma_2 \psi \Gamma\Gamma_1 \ \ (u) - u| < \delta \text{ for all } u \in \bar{\Omega}.$$

Hence by theorem 0.3 (iii) we have $d(\Gamma_2 \psi \Gamma\Gamma_1 \Lambda(u), \mathring{\Omega}, 0) = 1$. Therefore by theorem 0.3 (ii) there exists an element $u \in \mathring{\Omega}$ such that $\Gamma_2 \psi \Gamma\Gamma_1 \Lambda(u) = 0$. Setting $x^* = \Gamma_1\Lambda(u)$, we have $x^* \in V$ and $\psi\Gamma x^* = 0$. Obviously $\hat{x} = \Gamma x^*$ is a solution of the original MPBVP $Lx = Nx$ and $Q_m \hat{x} = x^*$. Since $\hat{x} \in \tilde{S}_o$, we clearly have $|||\hat{x} - x_o||| \leq d$ and $\mu(\hat{x} - x_o) \leq \bar{R}$. This completes the proof of the theorem.

<u>REMARK</u> 2.1. If X is of the form $X(t,x_o,\ldots,x_q)$ where $q \leq n-1$, then from our analysis of this chapter we observe that it is enough to consider the space $H^q(J)$. In this case $|||\cdot|||$ will be the corresponding norm on $H^q(J)$. Also, the quantities θ_m and $\bar{\theta}_m$ are to be defined as follows:

$$\theta_m = \sqrt{b-a}\ \Big(\sum_{i=0}^{q-1} \Big(\sup_{t \in J} \int_a^b \Big(\frac{\partial^i K_m(t,s)}{\partial t^i}\Big)^2 ds\Big)^{1/2}\Big) +$$

$$\Big(\int_a^b \int_a^b \Big(\frac{\partial^q K_m(t,s)}{\partial t^q}\Big)^2 ds\, dt\Big)^{1/2},$$

$$\bar{\theta}_m = \max_{i=0,\ldots,q} \Big(\sup_{t \in J} \int_a^b \Big(\frac{\partial^i K_m(t,s)}{\partial t^i}\Big)^2 ds\Big)^{1/2}.$$

Also, in this case μ will be the corresponding function on ~~$\tilde{H}^{q-1}(J)$~~. $\tilde{H}^q(J)$.

CHAPTER 3

AN ILLUSTRATIVE EXAMPLE

3.0. OUTLINE OF THE CHAPTER

In this chapter we make use of the theory developed in the earlier chapters and prove the existence of a solution to the following third order nonlinear three-point boundary value problem:

$$\begin{aligned} x^{(3)} &= (x\, x^{(1)})^2 + t - \frac{2}{\pi} \sin \pi t, \\ x^{(1)}(0) &= x^{(1)}(1) = x(\tfrac{1}{2}) = 0 \end{aligned} \tag{3.0.1}$$

over the interval $J = [0,1]$.

3.1. EXISTENCE OF A SOLUTION

Take $x = x^{(3)}$; $B_1(x) = x^{(1)}(0)$, $B_2(x) = x^{(1)}(1)$

$$\text{and} \quad B_3(x) = x(\tfrac{1}{2}). \tag{3.1.1}$$

As usual, the operator L is defined as follows:

$$\begin{aligned} D(L) &= \{ x \in H^3(J) : x^{(1)}(0) = x^{(1)}(1) = x(\tfrac{1}{2}) = 0 \}, \\ Lx &= x^{(3)}. \end{aligned} \tag{3.1.2}$$

First of all, we obtain the adjoint L^* of the operator L. Let $G^3(J)$ denote the space of all real-valued functions y on J where y is continuous and possesses first and second continuous derivatives on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $y^{(3)} \in S$.

At the point $t = \frac{1}{2}$, the function y and its first and second derivatives may have discontinuities of the first kind only.

Suppose $y \in G^3(J)$. Then

$$\int_0^1 x^{(3)} y \, dt = \int_0^1 y(dx^{(2)}) = \int_0^{1/2} y.\, d(x^{(2)}) + \int_{1/2}^1 y \, d(x^{(2)}). \tag{3.1.3}$$

Integrating by parts we have

$$\int_0^1 x^{(3)} y \, dt = yx^{(2)} \Big]_0^{\frac{1}{2}-0} - \int_0^{1/2} x^{(2)} y^{(1)} dt + yx^{(2)} \Big]_{\frac{1}{2}+0}^{1} - \int_{1/2}^1 x^{(2)} y^{(1)} dt$$

$$= y(\tfrac{1}{2}-0)x^{(2)}(\tfrac{1}{2}) - y(0)x^{(2)}(0) + y(1)x^{(2)}(1) - y(\tfrac{1}{2}+0)x^{(2)}(\tfrac{1}{2}) - \int_0^{1/2} x^{(2)} y^{(1)} dt - \int_{1/2}^1 x^{(2)} y^{(1)} dt. \tag{3.1.4}$$

Again,

$$\int_0^{1/2} x^{(2)} y^{(1)} dt + \int_{1/2}^1 x^{(2)} y^{(1)} dt = \int_0^{1/2} y^{(1)} d(x^{(1)}) + \int_{1/2}^1 y^{(1)} d(x^{(1)})$$

$$= x^{(1)} y^{(1)} \Big]_0^{\frac{1}{2}-0} + x^{(1)} y^{(1)} \Big]_{\frac{1}{2}+0}^{1} - \int_0^{1/2} y^{(2)} x^{(1)} dt - \int_{1/2}^1 y^{(2)} x^{(1)} dt$$

$$= x^{(1)}(\tfrac{1}{2}) y^{(1)}(\tfrac{1}{2}-0) - x^{(1)}(0) y^{(1)}(0) + x^{(1)}(1) y^{(1)}(1) - x^{(1)}(\tfrac{1}{2}) y^{(1)}(\tfrac{1}{2}+0) - \int_0^{1/2} y^{(2)} x^{(1)} dt - \int_{1/2}^1 y^{(2)} x^{(1)} dt. \tag{3.1.5}$$

Finally,

$$\int_0^{1/2} y^{(2)}x^{(1)}dt + \int_{1/2}^{1} y^{(2)}x^{(1)}dt = \int_0^{1/2} y^{(2)}d(x) + \int_{1/2}^{1} y^{(2)}d(x)$$

$$= y^{(2)}x\Big]_0^{\frac{1}{2}-0} + y^{(2)}x\Big]_{\frac{1}{2}+0}^{1} - \int_0^{1/2} x\, y^{(3)}dt - \int_{1/2}^{1} x\, y^{(3)}dt$$

$$= y^{(2)}(\tfrac{1}{2}-0)\ x(\tfrac{1}{2}) - y^{(2)}(0)\ x(0) + y^{(2)}(1)\ x(1)$$

$$- y^{(2)}(\tfrac{1}{2}+0)\ x(\tfrac{1}{2}) - \int_0^{1} xy^{(3)}dt. \tag{3.1.6}$$

Thus, from (3.1.4), (3.1.5) and (3.1.6), we get

$$\int_0^1 x^{(3)}y\ dt = -x(0)y^{(2)}(0)-y(0)x^{(2)}(0)+y(1)x^{(2)}(1)+x^{(1)}(0)y^{(1)}(0)$$

$$-x^{(1)}(1)y^{(1)}(1)+x(1)y^{(2)}(1)$$

$$+x(\tfrac{1}{2})\left[y^{(2)}(\tfrac{1}{2}-0) - y^{(2)}(\tfrac{1}{2}+0)\right]$$

$$+x^{(1)}(\tfrac{1}{2})\left[y^{(1)}(\tfrac{1}{2}+0) - y^{(1)}(\tfrac{1}{2}-0)\right]$$

$$+x^{(2)}(\tfrac{1}{2})\ \left[y(\tfrac{1}{2}-0) - y(\tfrac{1}{2}+0)\right]$$

$$+\int_0^1 x(-y^{(3)})dt. \tag{3.1.7}$$

We know that the formal adjoint of $\tau x = x^{(3)}$ is given by $\tau^* y = -y^{(3)}$.

Take $B_1^*(y) = y(0), B_2^*(y) = y(1), B_3^*(y) = y^{(2)}(0)$, $B_4^*(y) = y^{(2)}(1)$,

$B_5^*(y) = y^{(1)}(\frac{1}{2}+0) - y^{(1)}(\frac{1}{2}-0)$ and $B_6^*(y) = y(\frac{1}{2}-0)-y(\frac{1}{2}+0)$.

We observe, from (3.1.7), that the adjoint L^* of L is given by

$$D(L^*) = \{ y \in G^3(J) : y(0) = y(1) = y^{(2)}(0) = y^{(2)}(1) = 0,$$
$$y(\tfrac{1}{2}-0) - y(\tfrac{1}{2}+0) = 0,$$
$$y^{(1)}(\tfrac{1}{2}+0) - y^{(1)}(\tfrac{1}{2}-0) = 0 \},$$
$$L^*y = \tau^*y = -y^{(3)} . \tag{3.1.8}$$

Clearly, $N(L) = \emptyset$ and $N(L^*) = \emptyset$. Therefore, $p = \dim N(L) = 0$ and $q = \dim N(L^*) = 0$. Also the functions $\sqrt{2}\sin\pi kt$, $k=1,2,3,\ldots,m,\ldots\ldots$ belong to $D(L^*)$ and form a complete orthonormal set in S. The operator $T_1(\tau)$ is defined by $T_1(\tau)x = \tau x = x^{(3)}$ for all $x \in H^3(J)$. We can easily verify that the functions $1, \sqrt{3}(2t-1), \sqrt{5}\,(6t^2-6t+1)$ form an orthonormal basis for $N(T_1(\tau))$.

Take $\phi_1(t) = 1$, $\phi_2(t) = \sqrt{3}(2t-1)$ and $\phi_3(t) = \sqrt{5}(6t^2-6t+1)$. We have

$$\det \Phi(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) & \phi_3(t) \\ \phi_1^{(1)}(t) & \phi_2^{(1)}(t) & \phi_3^{(1)}(t) \\ \phi_1^{(2)}(t) & \phi_2^{(2)}(t) & \phi_3^{(2)}(t) \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{3}(2t-1) & \sqrt{5}(6t^2-6t+1) \\ 0 & 2\sqrt{3} & \sqrt{5}(12t-6) \\ 0 & 0 & 12\sqrt{5} \end{vmatrix}$$

$$= 24\sqrt{15},$$

$$W_1(t) = \begin{vmatrix} 0 & \phi_2(t) & \phi_3(t) \\ 0 & \phi_2^{(1)}(t) & \phi_3^{(1)}(t) \\ 1 & \phi_2^{(2)}(t) & \phi_3^{(2)}(t) \end{vmatrix} = \begin{vmatrix} 0 & \sqrt{3}(2t-1) & \sqrt{5}(6t^2-6t+1) \\ 0 & 2\sqrt{3} & \sqrt{5}(12t-6) \\ 1 & 0 & 12\sqrt{5} \end{vmatrix}$$

$$= 4\sqrt{15}\ (3t^2-3t+1),$$

$$W_2(t) = \begin{vmatrix} \phi_1(t) & 0 & \phi_3(t) \\ \phi_1^{(1)}(t) & 0 & \phi_3^{(1)}(t) \\ \phi_1^{(2)}(t) & 1 & \phi_3^{(2)}(t) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sqrt{5}(6t^2-6t+1) \\ 0 & 0 & \sqrt{5}(12t-6) \\ 0 & 1 & 12\sqrt{5} \end{vmatrix}$$

and

$$= -6\sqrt{5}(2t-1),$$

$$W_3(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) & 0 \\ \phi_1^{(1)}(t) & \phi_2^{(1)}(t) & 0 \\ \phi_1^{(2)}(t) & \phi_2^{(2)}(t) & 1 \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{3}(2t-1) & 0 \\ 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2\sqrt{3}.$$

Therefore,

$$G(t,s) = \sum_{i=1}^{3} \frac{\phi_i(t)\ W_i(s)}{\det \Phi(s)}$$

$$= (4\sqrt{15}\ (3s^2-3s+1) - 6\sqrt{15}(2t-1)(2s-1)$$

$$+ 2\sqrt{15}(6t^2-6t+1))/24\sqrt{15}.$$

By exact calculation we get

$$G(t,s) = \frac{1}{2}(t-s)^2. \tag{3.1.9}$$

<u>Determination of constants $A_{\ell j}$s:</u>

Let B be the matrix denoted by

$$B = (B_j(\phi_i)) = \begin{pmatrix} B_1(\phi_1) & B_1(\phi_2) & B_1(\phi_3) \\ B_2(\phi_1) & B_2(\phi_2) & B_2(\phi_3) \\ B_3(\phi_1) & B_3(\phi_2) & B_3(\phi_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 2\sqrt{3} & 6\sqrt{5} \\ 1 & 0 & -\sqrt{5}/2 \end{pmatrix}. \tag{3.1.10}$$

Calculating the inverse of B we get

$$A = (A_{\ell j}) = B^{-1} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{24} & \frac{1}{24} & 1 \\ \frac{1}{4\sqrt{3}} & \frac{1}{4\sqrt{3}} & 0 \\ -\frac{1}{12\sqrt{5}} & \frac{1}{12\sqrt{5}} & 0 \end{pmatrix}. \tag{3.1.11}$$

Thus $AB = \hat{I}$. (3.1.12)

Determination of ψ_ℓ s and $K(t,s)$:

Let $y \in R(L) = S$ and let $x = Hy$. Then

$$B_j(x) = \sum_{i=1}^{3} c_i B_j(\phi_i) + B_j\left(\int_0^t G(t,s)\, y(s)ds\right) = 0. \qquad (3.1.13)$$

But,

$$B_1\left(\int_0^t G(t,s)y(s)ds\right) = B_1\left(\int_0^t \frac{(t-s)^2}{2}\, y(s)ds\right)$$

$$= \left(\int_0^t \frac{(t-s)^2}{2}\, y(s)ds\right)^{(1)}(0)$$

$$= \left(\int_0^t (t-s)y(s)ds\right)(0) = 0,$$

$$B_2\left(\int_0^t G(t,s)y(s)ds\right) = B_2\left(\int_0^t \frac{(t-s)^2}{2}\, y(s)ds\right)$$

$$= \left(\int_0^t \frac{(t-s)^2}{2}\, y(s)ds\right)^{(1)}(1)$$

$$= \int_0^1 (1-s)\ y(s)\ ds,$$

$$B_3\left(\int_0^t G(t,s)y(s)ds\right) = B_3\left(\int_0^t \frac{(t-s)^2}{2}\, y(s)ds\right)$$

$$= \int_0^{1/2} \frac{(s-\frac{1}{2})^2}{2}\ y(s)ds$$

$$= \int_0^{1/2} \frac{(2s-1)^2}{8}\, y(s)ds.$$

Therefore, from (3.1.13), we get

$$\sum_{i=1}^{3} c_i B_1(\phi_i) = 0, \tag{3.1.14}$$

$$\sum_{i=1}^{3} c_i B_2(\phi_i) + \int_0^1 (1-s)y(s)ds = 0, \tag{3.1.15}$$

and

$$\sum_{i=1}^{3} c_i B_3(\phi_i) + \int_0^{1/2} \frac{(2s-1)^2}{8} y(s)ds = 0. \tag{3.1.16}$$

Multiplying (3.1.14) with $A_{\ell 1}$, (3.1.15) with $A_{\ell 2}$ and (3.1.16) with $A_{\ell 3}$ and adding them we get

$$\sum_{i=1}^{3} c_i \left[A_{\ell 1}B_1(\phi_i) + A_{\ell 2}B_2(\phi_i) + A_{\ell 3}B_3(\phi_i) \right]$$

$$+ A_{\ell 2} \int_0^1 (1-s)y(s)ds + A_{\ell 3} \int_0^{1/2} \frac{(2s-1)^2}{8} y(s)ds = 0. \tag{3.1.17}$$

Taking $\ell = 1$, from (3.1.17) and (3.1.12), we have

$$c_1 = -A_{12} \int_0^1 (1-s)y(s)ds - A_{13} \int_0^{1/2} \frac{(2s-1)^2}{8} y(s)ds$$

$$= -\frac{1}{24} \int_0^1 (1-s)y(s)ds - \int_0^{1/2} \frac{(2s-1)^2}{8} y(s)ds \quad (\text{see}(3.1.11))$$

$$= -\int_0^{1/2} \left[\frac{(1-s)}{24} + \frac{(2s-1)^2}{8} \right] y(s)ds$$

$$- \frac{1}{24} \int_{1/2}^{1} (1-s)\, y(s)ds.$$

Therefore,

$$\psi_1(s) = -\left[\frac{(1-s)}{24} + \frac{(2s-1)^2}{8} \right] \chi_{[0,1/2)}(s)$$

$$- \frac{(1-s)}{24} \quad \chi_{(1/2,1]}(s)$$

$$= \left[-\frac{1}{2} s^2 - \frac{13}{24} s + \frac{1}{6}\right] \chi_{[0,1/2)}(s) - \frac{(1-s)}{24} \chi_{(1/2,1]}(s). \tag{3.1.18}$$

Similarly, for $\ell = 2$, from (3.1.17) and (3.1.12) we get

$$c_2 = -A_{22} \int_0^1 (1-s)y(s)ds - A_{23} \int_0^{1/2} \frac{(2s-1)^2}{8} y(s)ds$$

$$= -\frac{1}{4\sqrt{3}} \int_0^1 (1-s)y(s)ds. \qquad \text{(see (3.1.11))}.$$

Thus we have

$$\psi_2(s) = -\frac{(1-s)}{4\sqrt{3}}. \tag{3.1.19}$$

In a similar manner we obtain

$$\psi_3(s) = -\frac{(1-s)}{12\sqrt{5}}. \tag{3.1.20}$$

Having obtained the expressions for ψ_ℓs, we now derive below an explicit expression for $K(t,s)$.

$$\sum_{\ell=1}^{3} \phi_\ell(t)\,\psi_\ell(s) = -\left(\frac{1}{2} s^2 - \frac{13}{24} s + \frac{1}{6}\right) \chi_{[0,1/2)}(s)$$

$$- \frac{(1-s)}{24} \chi_{(1/2,1]}(s) - \frac{\sqrt{3}}{4\sqrt{3}} (2t-1)(1-s)$$

$$- \frac{\sqrt{5}}{12\sqrt{5}} (6t^2-6t+1)\,(1-s).$$

Then, after a simple calculation, we get

$$K(t,s) = \begin{cases} -(\frac{1}{2}s^2 - \frac{13}{24}s + \frac{1}{6})\chi_{[0,1/2)}(s) - \frac{(1-s)}{24}\chi_{(1/2,1]}(s) \\ \qquad - (1-s)(\frac{1}{2}t^2 - \frac{1}{6}) + \frac{(t-s)^2}{2} \quad \text{for } 0 \le s \le t \le 1, \\ -(\frac{1}{2}s^2 - \frac{13}{24}s + \frac{1}{6})\chi_{[0,1/2)}(s) - \frac{(1-s)}{24}\chi_{(1/2,1]}(s) \\ \qquad - (1-s)(\frac{1}{2}t^2 - \frac{1}{6}) \quad \text{for } 0 \le t \le s \le 1. \end{cases} \tag{3.1.21}$$

<u>Determination of $K_1(t,s)$:</u>

We take $\omega_1 = \sqrt{2}\sin\pi t \in D(L^*)$. We define

$P_1(x) = (x, \sqrt{2}\sin\pi(\cdot))\sqrt{2}\sin\pi(\cdot)$.

Since $K_1(t,s) = K(t,s) - \sqrt{2}(\int_0^1 K(t,\xi)\sin\pi\xi\, d\xi)\sqrt{2}\sin\pi s$, we obtain a more suitable expression for $\int_0^1 K_1(t,s)\sin\pi s\, ds$. Obviously,

$$\int_0^1 K(t,s)\sin\pi s\, ds = \begin{cases} \int_0^t K(t,s)\sin\pi s\, ds + \int_t^{1/2} K(t,s)\sin\pi s\, ds \\ \qquad + \int_{1/2}^1 K(t,s)\sin\pi s\, ds \quad \text{for } t \le 1/2, \\ \int_0^{1/2} K(t,s)\sin\pi s\, ds + \int_{1/2}^t K(t,s)\sin\pi s\, ds \\ \qquad + \int_{t}^1 K(t,s)\sin\pi s\, ds \quad \text{for } t \ge 1/2. \end{cases}$$

For $0 \le t \le 1/2$, we have

$$\int_0^t K(t,s)\sin\pi s\, ds = \int_0^t \Big[-(\frac{1}{2}s^2 - \frac{13}{24}s + \frac{1}{6}) - (1-s)(\frac{1}{2}t^2 - \frac{1}{6}) + \frac{(t-s)^2}{2}\Big]\sin\pi s\, ds,$$

$$\int_t^{1/2} K(t,s)\sin\pi s ds = \int_t^{1/2} \Big[-(\tfrac{1}{2}s^2 - \tfrac{13}{24}s + \tfrac{1}{6}) - (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6})\Big]\sin\pi s\, ds,$$

$$\int_{1/2}^1 K(t,s)\sin\pi s\, ds = \int_{1/2}^1 -\Big[\frac{(1-s)}{24} + (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6})\Big]\sin\pi s\, ds.$$

Therefore,

$$\int_0^1 K(t,s)\sin\pi s\, ds = \int_0^{1/2} -(\tfrac{1}{2}s^2 - \tfrac{13}{24}s + \tfrac{1}{6})\sin\pi s\, ds$$

$$- \int_0^1 (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6})\sin\pi s\, ds - \int_{1/2}^1 \frac{(1-s)}{24}\sin\pi s ds$$

$$+ \int_0^t \frac{(t-s)^2}{2}\sin\pi s\, ds \quad \text{for } t \leq 1/2.$$

Similarly, for $t \geq 1/2$ we have

$$\int_0^{1/2} K(t,s)\sin\pi s\, ds = \int_0^{1/2} \Big[-(\tfrac{1}{2}s^2 - \tfrac{13}{24}s + \tfrac{1}{6}) - (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6}) + \frac{(t-s)^2}{2}\Big]\sin\pi s\, ds,$$

$$\int_{1/2}^t K(t,s)\sin\pi s\, ds = \int_{1/2}^t \Big[-\frac{(1-s)}{24} - (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6}) + \frac{(t-s)^2}{2}\Big]\sin\pi s\, ds,$$

$$\int_t^1 K(t,s)\sin\pi s\, ds = \int_t^1 \Big[-\frac{(1-s)}{24} - (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6})\Big]\sin\pi s\, ds.$$

So,

$$\int_0^1 K(t,s)\sin\pi s\, ds = \int_0^{1/2} -(\tfrac{1}{2}s^2 - \tfrac{13}{24}s + \tfrac{1}{6})\sin\pi s\, ds$$

$$- \int_0^1 (1-s)(\tfrac{1}{2}t^2 - \tfrac{1}{6})\sin\pi s\, ds$$

$$- \int_{1/2}^1 \frac{(1-s)}{24}\sin\pi s\, ds + \int_0^t \frac{(t-s)^2}{2}\sin\pi s\, ds$$

$$\text{for } t \geq 1/2.$$

Noting that the expressions for $\int_0^1 K(t,s)\sin\pi s \, ds$ in both the cases $t \in [0,1/2]$ and $t \in [1/2,1]$ are the same, for all $t \in J$ we have

$$\int_0^1 K(t,s)\sin\pi s \, ds = -\int_0^{1/2} \left(\tfrac{1}{2}\, s^2 - \tfrac{13}{24}\, s + \tfrac{1}{6}\right) \sin\pi s \, ds$$
$$- \int_0^1 (1-s)\left(\tfrac{1}{2}\, t^2 - \tfrac{1}{6}\right) \sin\pi s \, ds$$
$$- \int_{1/2}^1 \frac{(1-s)}{24} \sin\pi s \, ds + \int_0^t \frac{(t-s)^2}{2} \sin\pi s \, ds. \tag{3.1.22}$$

By elementary integration, we get

$$\int_0^1 K(t,s)\sin\pi s \, ds = -\frac{t^2}{2\pi} + \frac{1}{\pi^3} + \int_0^t \frac{(s^2-2ts+t^2)}{2} \sin\pi s \, ds. \tag{3.1.23}$$

But,

$$\int_0^t \frac{(s^2-2st+t^2)}{2} \sin\pi s \, ds = -\frac{1}{\pi^3} + \frac{t^2}{2\pi} + \frac{\cos \pi t}{\pi^3}. \tag{3.1.24}$$

Therefore, from (3.1.23) and (3.1.24), we get

$$\int_0^1 K(t,s) \sin\pi s \, ds = \frac{\cos\pi t}{\pi^3}. \tag{3.1.25}$$

Thus we get the following expression for $K_1(t,s)$:

$$K_1(t,s) = K(t,s) - \frac{2\cos\pi t \sin\pi s}{\pi^3}.$$

Substituting the expression for $K(t,s)$, we get

$$K_1(t,s)=\begin{cases} -(\frac{1}{2}s^2-\frac{13}{24}s+\frac{1}{6})\chi_{[0,1/2)}(s)-\frac{(1-s)}{24}\chi_{(1/2,1]}(s) \\ \quad -(1-s)(\frac{1}{2}t^2-\frac{1}{6})+\frac{(t-s)^2}{2}-\frac{2}{\pi^3}\cos\pi t\sin\pi s \\ \qquad\qquad \text{for } 0\le s\le t\le 1, \\ -(\frac{1}{2}s^2-\frac{13}{24}s+\frac{1}{6})\chi_{[0,1/2)}(s)-\frac{(1-s)}{24}\chi_{(1/2,1]}(s) \\ \quad -(1-s)(\frac{1}{2}t^2-\frac{1}{6})-\frac{2\cos\pi t\sin\pi s}{\pi^3} \text{ for } 0\le t\le s\le t. \end{cases} \tag{3.1.26}$$

Determination of θ_1 and $\bar{\theta}_1$:

First of all, we have

$$\int_0^1 K_1(t,s)^2 ds = \begin{cases} \int_0^t K_1(t,s)^2 ds+\int_t^{1/2} K_1(t,s)^2 ds+\int_{1/2}^1 K_1(t,s)^2 ds \\ \qquad\qquad \text{for } t\le 1/2, \\ \int_0^{1/2} K_1(t,s)^2 ds+\int_{1/2}^t K_1(t,s)^2 ds \\ \qquad +\int_t^1 K_1(t,s)^2 ds \quad \text{for } t\ge 1/2. \end{cases}$$

Substituting the expression for $K_1(t,s)$ on each of the subintervals, elementary and lengthy calculations yields that

$$\int_0^1 K_1(t,s)^2 ds = \begin{cases} \frac{t^6}{24}-\frac{2}{15}t^5+\frac{11}{96}t^4-\frac{3}{128}t^2+\frac{17}{7680}-\frac{2}{\pi^6}\cos^2\pi t \\ \qquad\qquad \text{for } 0\le t\le 1/2, \\ \frac{t^6}{24}-\frac{7}{60}t^5+\frac{7}{96}t^4+\frac{1}{24}t^3-\frac{17}{384}t^2 \\ \quad +\frac{1}{192}t+\frac{13}{7680}-\frac{2}{\pi^6}\cos^2\pi t \quad \text{for } 1/2\le t\le 1. \end{cases} \tag{3.1.27}$$

We can check that $\int_0^1 K_1(t,s)^2 ds$ is continuous on $J = [0,1]$. Since $\int_0^1 K_1(t,s)^2 ds \geq 0$ for $t \in J$ and $\cos^2 \pi t \geq 0$ for $t \in J$, from (3.1.27) we get

$$\int_0^1 K_1(t,s)^2 ds \leq \begin{cases} \frac{t^6}{24} - \frac{2}{15} t^5 + \frac{11}{96} t^4 - \frac{3}{128} t^2 + \frac{17}{7680} & \text{for } 0 \leq t \leq 1/2, \\ \frac{t^6}{24} - \frac{7}{60} t^5 + \frac{7}{96} t^4 + \frac{1}{24} t^3 - \frac{17}{384} t^2 + \frac{1}{192} t + \frac{13}{7680} & \text{for } 1/2 \leq t \leq 1. \end{cases} \tag{3.1.28}$$

Take

$$\alpha_1(t) = \frac{t^6}{24} - \frac{2}{15} t^5 + \frac{11}{96} t^4 - \frac{3}{128} t^2 + \frac{17}{7680} . \tag{3.1.29}$$

Then $\alpha_1^{(1)}(t) = \frac{t^5}{4} - \frac{2}{3} t^4 + \frac{11}{24} t^3 - \frac{3}{64} t$.

The points $t = 0,\ 1/2,\ 3/2,\ \frac{1}{6}(2 \pm \sqrt{13})$ are solutions of the equation $\alpha_1^{(1)}(t) = 0$. We notice that $t = 3/2,\ \frac{1}{6}(2 \pm \sqrt{13})$ lie strictly outside the interval $[0,1/2]$. Moreover, $\alpha_1(0) = \frac{17}{7680}$ and $\alpha_1(1/2) = 0$. Therefore,

$$\sup_{t \in [0,1/2]} \alpha_1(t) = \frac{17}{7680} . \tag{3.1.30}$$

Take

$$\alpha_2(t) = \frac{t^6}{24} - \frac{7}{60} t^5 + \frac{7}{96} t^4 + \frac{1}{24} t^3 - \frac{17}{384} t^2 + \frac{1}{192} t + \frac{13}{7680} . \tag{3.1.31}$$

Then $\alpha_2^{(1)}(t) = \frac{t^5}{4} - \frac{7}{12} t^4 + \frac{7}{24} t^3 + \frac{1}{8} t^2 - \frac{17}{192} t + \frac{1}{192}$.

The points $t = -1/2, 1/2, 1, \frac{2}{3}\ (1 \pm \sqrt{\frac{13}{4}})$ are solutions of the equation $\alpha_2^{(1)}(t) = 0$. Moreover, the points $t = -1/2,\ \frac{2}{3}(1 \pm \sqrt{\frac{13}{4}})$ lie strictly outside the interval $[1/2, 1]$. Also, $\alpha_2(1) = \frac{17}{7680}$ and $\alpha_2(\frac{1}{2}) = 0$. Therefore,

$$\sup_{t \in [1/2, 1]} \alpha_2(t) = \frac{17}{7680}. \tag{3.1.32}$$

Hence, from (3.1.28), (3.1.29), (3.1.30), (3.1.31) and (3.1.32), we get

$$\sup_{t \in [0,1]} \int_0^1 K_1(t,s)^2 ds \leq \frac{17}{7680}.$$

Therefore,

$$\left(\sup_{t \in [0,1]} \int_0^1 K_1(t,s)^2 ds \right)^{1/2} \leq \sqrt{\frac{17}{7680}} < 0.047049. \tag{3.1.33}$$

Differentiating the expression (3.1.26) with respect to t, we get

$$\frac{\partial K_1(t,s)}{\partial t} = \begin{cases} -(1-s)t + \frac{2}{\pi^2} \sin\pi t \sin\pi s + (t-s) & \text{for } 0 \leq s \leq t \leq 1, \\ -(1-s)t + \frac{2}{\pi^2} \sin\pi t \sin\pi s & \text{for } 0 \leq t \leq s \leq 1. \end{cases}$$

$$= \begin{cases} s(t-1) + \frac{2}{\pi^2} \sin\pi t \sin\pi s & \text{for } 0 \leq s \leq t \leq 1, \\ t(s-1) + \frac{2}{\pi^2} \sin\pi t \sin\pi s & \text{for} \leq t \leq s \leq 1. \end{cases} \tag{3.1.34}$$

Therefore,

$$\int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds = \int_0^t \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds + \int_t^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds$$

$$= \int_0^t \left[s(t-1) + \frac{2}{\pi^2} \sin\pi t \sin\pi s\right]^2 ds$$

$$+ \int_t^1 \left[t(s-1) + \frac{2}{\pi^2} \sin\pi t \sin\pi s\right]^2 ds.$$

After elementary and lengthy integrations, we get

$$\int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds = \frac{1}{3} t^4 - \frac{2}{3} t^3 + \frac{1}{3} t^2 - \frac{2}{\pi^4} \sin^2 \pi t, \ t \in J. \tag{3.1.35}$$

Since $\int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds \geq 0$ for $t \in J$ and $\sin^2 \pi t \geq 0$ for $t \in J$, from (3.1.35) we get

$$\int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds \leq \frac{1}{3} t^4 - \frac{2}{3} t^3 + \frac{1}{3} t^2, \ t \in J. \tag{3.1.36}$$

Take $\quad \beta(t) = \frac{1}{3} t^4 - \frac{2}{3} t^3 + \frac{1}{3} t^2.$

Then $\quad \beta^{(1)}(t) = \frac{4}{3} t^3 - 2t^2 + \frac{2}{3} t.$

The points $t = 0, 1/2, 1$ are solutions of the equation $\beta^{(1)}(t) = 0$. Moreover, $\beta(0) = 0, \beta(\frac{1}{2}) = \frac{1}{48}$ and $\beta(1) = 0$. Thus

$$\sup_{t \in [0,1]} \beta(t) = \frac{1}{48}. \tag{3.1.37}$$

Therefore, from (3.1.36) and (3.1.37), we have

$$\left(\sup_{t \in [0,1]} \int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\right)^{1/2} \leq \frac{1}{\sqrt{48}} < 0.1443377. \tag{3.1.38}$$

Also, a simple integration of (3.1.35) yields that

$$\int_0^1 \int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\, dt = \frac{1}{90} - \frac{1}{\pi^4}.$$

Hence,

$$\left(\int_0^1 \int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\ dt\right)^{1/2} = \left(\frac{1}{90} - \frac{1}{\pi^4}\right)^{\frac{1}{2}} < 0.029496. \qquad (3.1.39)$$

From (3.1.33) and (3.1.39), we get

$$\theta_1 = \left(\sup_{t \in [0,1]} \int_0^1 K_1(t,s)^2 ds\right)^{1/2} + \left(\int_0^1 \int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds dt\right)^{1/2}$$

$$< 0.047049 + 0.029496$$

$$= 0.076545.$$

Therefore,

$$\theta_1 < 0.076545. \qquad (3.1.40)$$

Also, from (3.1.33) and (3.1.38), we get

$$\bar{\theta}_1 = \max\left(\left(\sup_{t \in [0,1]} \int_0^1 K_1(t,s)^2 ds\right)^{1/2}, \left(\sup_{t \in [0,1]} \int_0^1 \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\right)^{1/2}\right)$$

$$< \max(0.047049,\ 0.1443377)$$

$$= 0.1443377.$$

Therefore,

$$\bar{\theta}_1 < 0.1443377. \qquad (3.1.41)$$

Here $X(t,x,x^{(1)}) = (xx^{(1)})^2 + t - \frac{2}{\pi}\sin\pi t$. Therefore, $(Nx)(t) = (x(t)x^{(1)}(t))^2 + t - \frac{2}{\pi}\sin\pi t$.

Take $m = 1$.

Take $\eta_1 = \sqrt{2}\,\cos\pi(\cdot) \in D(L)$. Let $S_o = \langle \sqrt{2}\,\cos\pi(\cdot) \rangle$. We have $\bar{m} = p+m-q=m=1$. The functions $r_1 : E^1 \to S_o$, $r_2 : \langle \sqrt{2}\,\sin\pi(\cdot) \rangle \to E'$ are given by

$$r_1(\xi) = \sqrt{2}\,\xi\,\cos\pi(\cdot),\ \xi \in E^1, \tag{3.1.42}$$

and

$$r_2\,(\sqrt{2}u\,\sin\pi(\cdot)) = u. \tag{3.1.43}$$

Let $\xi \in E^1$ and consider $r_1(\xi) = \xi\sqrt{2}\,\cos\pi(\cdot) = x \in S_o$.

Then

$$L\,r_1(\xi) = (\sqrt{2}\,\xi\,\cos\pi(\cdot))^{(3)} = \sqrt{2}\,\xi\,\pi^3\sin\pi(\cdot).$$

Hence,

$$P_1 L\,r_1(\xi) = \sqrt{2}\,\xi\,\pi^3\,\sin\pi(\cdot). \tag{3.1.44}$$

Also, we get

$$(N\,r_1(\xi))(t) = \xi^4\,\pi^2\,\sin^2 2\pi t + t - \frac{2}{\pi}\sin\pi t.$$

Therefore,

$$(P_1 N\,r_1(\xi)) = \xi^4\,\pi^2 P_1(\sin^2 2\pi(\cdot)) + P_1(I_1(\cdot) - \frac{2}{\pi}\sin\pi(\cdot)).$$

After simple calculations, we get

$$(P_1 N\,r_1(\xi))(t) = \frac{32}{15}\,\pi\,\xi^4\,\sin\pi t. \tag{3.1.45}$$

From (3.1.44) and (3.1.45), we have

$$\begin{aligned}\psi\,r_1(\xi) &= P_1(Lr_1(\xi) - N\,r_1(\xi)) \\ &= \sqrt{2}\,\xi\,\pi^3\,\sin\pi(\cdot) - \frac{32}{15}\,\pi\,\xi^4\,\sin\pi(\cdot).\end{aligned} \tag{3.1.46}$$

We know that

$$\Psi(\xi) = \Gamma_2 \psi\, \Gamma_1(\xi), \ \xi \in E^1. \tag{3.1.47}$$

Hence, from (3.1.46) and (3.1.43) we get

$$\Psi(\xi) = \xi \pi^3 - \frac{16\sqrt{2}}{15} \pi \xi^4. \tag{3.1.48}$$

We clearly notice that $\xi_0 = 0$ is a solution if $\Psi(\xi) = 0$.

Take $x_0 = 0 \in S_0$. We notice that $(Nx_0)(t) = t - \frac{2}{\pi} \sin\pi t$.

Calculating, we get

$$\begin{aligned} ||Nx_0||^2 &= ||I_1(\cdot) - \frac{2}{\pi} \sin\pi(\cdot)||^2 \\ &= \frac{1}{3} - \frac{49}{242}. \end{aligned}$$

Hence,

$$||Nx_0|| = \left(\frac{1}{3} - \frac{49}{242}\right)^{1/2} < 0.36318. \tag{3.1.49}$$

Therefore,

$$\begin{aligned} |||H(I-P_1)Nx_0||| &\leq \theta_1 ||Nx_0|| \\ &\leq (0.076545) \times (0.36318) \quad \text{(by (3.1.40) and (3.1.49))} \\ &\leq 0.0277997. \end{aligned}$$

Thus,

$$|||H(I-P_1)Nx_0||| < e = 0.0277997. \tag{3.1.50}$$

Also,

$$\begin{aligned} \mu(H(I-P_1)Nx_0) &\leq \bar{\theta}_1 ||Nx_0|| \\ &\leq (0.1443377) \times (0.36318) \quad \text{(by (3.1.41) and (3.1.49))} \\ &< 0.0524206. \end{aligned}$$

Thus,

$$\mu(H(I-P_1)Nx_o) < \bar{e} = 0.0524206. \tag{3.1.51}$$

Take $x \in S_o$. Then $x = \xi\sqrt{2}\cos\pi(.)$. Therefore,

$$|||x||| = \sup_{t \in [0,1]} |x(t)| + ||x^{(1)}||$$

$$= |\xi|\sqrt{2} + ||-\pi\xi\sqrt{2}\sin\pi(.)||$$

$$= |\xi|\sqrt{2} + \pi|\xi| = (\sqrt{2} + \pi)\,|\xi|.$$

Hence,

$$|||x-x_o||| = ||\,x||| = (\sqrt{2} + \pi)\,|\xi|. \tag{3.1.52}$$

Also,

$$\mu(x) = \max\left(\sup_{t \in [0,1]} |\xi\sqrt{2}\cos\pi t|,\ \sup_{t \in [0,1]} |-\pi\xi\sqrt{2}\sin\pi\ \ |\right)$$

$$= \max(|\xi|\sqrt{2},\ \pi\sqrt{2}\,|\xi|)$$

$$= \pi\sqrt{2}\,|\xi|.$$

Therefore,

$$\mu(x-x_o) = \mu(x) = \pi\sqrt{2}\,|\xi|. \tag{3.1.53}$$

Suppose $|||x||| \leq c$ for some $c > 0$. Then from (3.1.52) we have $(\sqrt{2} + \pi)\,|\xi| \leq c$. Therefore,

$$|\xi| \leq \frac{c}{(\sqrt{2} + \pi)}.$$

Then, from (3.1.53), we have

$$\mu(x) \le \frac{c\pi\sqrt{2}}{(\sqrt{2} + \pi)} .$$

Take the sets

$$U = \{ \xi \in E^1 : |\xi - \xi_o| = |\xi| \le \frac{c}{(\sqrt{2}+\pi)} \}, \tag{3.1.54}$$

and

$$V = \{ x \in S_o : |||x - x_o||| = |||x||| \le c, \quad \mu(x) \le \frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} \} . \tag{3.1.55}$$

We clearly observe that the map $\Gamma_1 : E^1 \to S_o$ maps the set U into V. We take

$$\epsilon = \frac{c}{(\sqrt{2}+\pi)} ,$$

and (3.1.56)

$$r = \frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} .$$

We note that

$$\begin{aligned} Nx - Ny &= (xx^{(1)})^2 - (yy^{(1)})^2 \\ &= (xx^{(1)} + yy^{(1)}) (xx^{(1)} - yy^{(1)}) . \end{aligned}$$

Therefore,

$$|Nx - Ny| \le (|x||x^{(1)}| + |y||y^{(1)}|)(|x||x^{(1)} - y^{(1)}| + |y^{(1)}||x - y|) .$$

Thus for $x, y \in \tilde{S}_o$, we have

$$\begin{aligned} |Nx - Ny| &\le 2\bar{R}^2 \left[\bar{R} \, |x - y| + \bar{R} \, |x^{(1)} - y^{(1)}| \right] \\ &= 2\bar{R}^3 \left[|x - y| + |x^{(1)} - y^{(1)}| \right] . \end{aligned}$$

Let us take $\quad k_o = 2\bar{R}^3$.

By remark 2.1, the conditions of theorem 2.4 are equivalent to

$$\theta_1 k_o < (0.076545)\times(2\bar{R}^3) < 1,\ 0 < c < d,\ \frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} < \bar{R},$$

$$c+e \leq c+0.0277997 \leq (1-0.15309\,\bar{R}^3)\,d \leq (1-\theta_1 k_o)d,$$

$$\frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} + \bar{e} \leq \frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} + 0.0524206 \leq \bar{R} - 0.2886754\,\bar{R}^3 d$$

$$\leq \bar{R} - \bar{\theta}_1 k_o d,$$

$$(\theta_1 k_o d+e)k_o < (0.15309\,\bar{R}^3 d + 0.0277997)\,2\bar{R}^3 \leq \delta,$$

and there should exist $\xi_1, \xi_2 \in U$ such that

$$\Psi(\xi_1) \geq \delta \quad \text{and} \quad \Psi(\xi_2) \leq -\delta \quad \text{where} \quad \delta > 0 \quad \text{is some number.}$$

We shall find constants $c, d, \bar{R}, \delta$ and $\xi_1, \xi_2 \in U$ such that the following inequalities are satisfied.

$$0.15309\,\bar{R}^3 < 1,\ 0 < c < d,\ \frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} < c < \bar{R},$$

$$c+0.0277997 \leq (1-0.15309\,\bar{R}^3)d,$$

$$\frac{c\pi\sqrt{2}}{(\sqrt{2}+\pi)} + 0.0524206 \leq c + 0.0524206 \leq \bar{R} - 0.2886754\,\bar{R}^3 d,$$

$$(0.30618\,\bar{R}^3 d + 0.0555994)\,\bar{R}^3 \leq \delta,$$

and

$$\Psi(\xi_1) = \xi_1\pi^3 - \frac{16\sqrt{2}}{15}\pi\,\xi_1^4 \geq \delta,$$

$$\Psi(\xi_2) = \xi_2\pi^3 - \frac{16\sqrt{2}}{15}\pi\,\xi_2^4 \leq -\delta.$$

One possible choice for the above quantities is

$c=0.01$, $d=0.2$, $\bar{R}=0.1$, $\delta=0.00006$, $\xi_1 = \dfrac{0.01}{(\sqrt{2}+\pi)}$ and $\xi_2 = \dfrac{-0.01}{(\sqrt{2}+\pi)}$.

We can easily check that for this choice the above relation are satisfied. Indeed, for this choice it is easily to verify the above relations except the last two and the last two inequalities become

$$\xi_1\pi^3 - \frac{16\sqrt{2}}{15}\pi\xi_1^4 > 0.01 > \delta = 0.00006,$$

and

$$\xi_2\pi^3 - \frac{16\sqrt{2}}{15}\pi\xi_2^4 < -0.01 < -\delta = -0.00006.$$

Obviously, for this choice of the quantities the first mentioned relations are also satisfied. Hence, by remark 2.1 and theorem 2.4, the MPBVP

$$x^{(3)} = (xx^{(1)})^2 + t - \frac{2}{\pi}\sin\pi t,$$

$$x^{(1)}(0) = x^{(1)}(1) = x(\tfrac{1}{2}) = 0$$

has a solution $\hat{x}$ over the interval $[0,1]$. Moreover,

$$|\hat{x}(t)| \leq \theta|_{0.1} \text{ and } |\hat{x}^{(1)}(t)| \leq 0.1 \text{ for all } t \in [0,1].$$

CHAPTER 4

EXISTENCE OF AN ISOLATED SOLUTION OF A MULTI-POINT BOUNDARY VALUE PROBLEM

4.0. OUTLINE OF THE CHAPTER

In this chapter we follow the notations of Chapter 1 and find the existence of an isolated solution to the following MPBVP:

$$Lx = Nx \tag{4.0.1}$$

over the interval J where N is defined subsequently and L is the operator defined by (1.1.3).

4.1. NOTATIONS AND ASSUMPTIONS

Let R_i, $i=0,\ldots,n-1$ be positive real numbers. We denote

$$D = \{(x_o,\ldots,x_{n-1}) \in R^n: |x_i| \leq R_i,\ i=0,\ldots,n-1\} .$$

Let $X(t,x_o,\ldots,x_{n-1})$ be a nonlinear real-valued function defined on $J \times D$. We assume that $X(t,x_o,\ldots,x_{n-1})$ is continuously differentiable with respect to $(x_o,\ldots,x_{n-1})$ in $J \times D$. Further, we assume that $X(\cdot,x_o,\ldots,x_{n-1})$ is continuous for each fixed $(x_o,\ldots,x_{n-1}) \in D$. We define the operator N as follows:

$$D(N) = \{x \in \tilde{H}^{n-1}(J): \sup_{t \in J} |x^{(i)}(t)| \leq R_i,\ i=0,\ldots,n-2,$$
$$\text{ess.} \sup_{t \in J} |x^{(n-1)}(t)| \leq R_{n-1}\} , \tag{4.1.1}$$

$$(Nx)(t) = X(t,x(t),\ldots,x^{(n-1)}(t)) \text{ for all } t \in J \text{ for which } |x^{(n-1)}(t)| \leq R_{n-1}.$$

We shall find an isolated solution of the MPBVP

$$Lx = Nx \tag{4.1.2}$$

where N is defined by (4.1.1). From now onwards we take N to be the operator given by (4.1.1).

4.2. EXISTENCE OF AN ISOLATED SOLUTION

Let τ_o be a formal differential operator of order n-1 given by

$$\tau_o x = q_{n-1}(t) \frac{d^{n-1}x}{dt^{n-1}} + q_{n-2}(t) \frac{d^{n-2}x}{dt^{n-2}} + \cdots + q_o(t)x \tag{4.2.1}$$

where $q_i \in C^\infty(J)$, $i=0,\ldots,n-1$.

Let L_o be the differential operator generated by $\tau - \tau_o$ and B_js as in (1.1.3) where τ is given by (1.1.1) and B_js are given by (1.1.2). We remember that $B_1,\ldots,B_k$, $k \leq n$ are linearly independent. Let $K_o(t,s)$ be the function corresponding to L_o as in (1.3.10).

We now state the following lemma

<u>LEMMA</u> 4.1. If '0' is not an eigenvalue of the operator L_o, then the MPBVP

$$L_o x = \phi \; , \; \phi \in S \tag{4.2.2}$$

has one and only one solution on J given by

$$x(t) = \int_a^b K_o(t,s) \; \phi(s)ds. \tag{4.2.3}$$

Proof: Since zero is not an eigenvalue of the operator L_o, we observe that $\dim N(L_o) = 0$, and hence $\dim N(L_o^*) = 0$ (see 1.1(v)). Moreover, since $S = R(L_o) \oplus N(L_o^*)$, we get $S = R(L_o)$. Denoting H_o to be the inverse of L_o, we get as in (1.3.11) that

$$(H_o \phi)(t) = \int_a^b K_o(t,s)\, \phi(s) ds, \quad t \in J$$

for $\phi \in S$. Obviously, this is the solution of (4.2.2) on J. This completes the proof of the lemma.

We say that $\bar{x} \in D(L) \cap D(N)$ is an approximate solution of (4.1.2) if $\bar{x}$ satisfies the equation $\tau x = Nx$ approximately.

The following theorem gives the existence of an isolated solution of (4.1.2).

THEOREM 4.1. Let the assumptions of section 4.1 be satisfied. Assume that the MPBVP (4.1.2) has an approximate solution $\bar{x} \in D(L) \cap D(N)$. Let τ_o be the operator given by (4.2.1). Suppose there exist a positive constant δ and a non-negative constant $\rho < 1$ such that

(i) '0' is not an eigenvalue of L_o,

(ii) $$D_\delta = \{(x_o,\ldots,x_{n-1}) \in R^n : |x_i - \bar{x}^{(i)}(t)| \le \delta \text{ for some } t \in J,\ i=0,\ldots,n-1\} \subset D, \tag{4.2.4}$$

(iii) $$\sum_{i=0}^{n-1} \left| \frac{\partial X(t,x_o,\ldots,x_{n-1})}{\partial x_i} - q_i(t) \right| \le \frac{\rho}{M} \text{ for all } (x_o,\ldots,x_{n-1})$$ such that $|x_i - \bar{x}^{(i)}(t)| \le \delta$, and $t \in J$,

(iv) $\frac{\tilde{M}r}{1-\rho} \leq \delta$.

Here L_0 is the operator generated by $\tau - \tau_0$ and B_js; $\tilde{M} = \max_{i=0,\ldots,n-1} M_i$ where M_is are positive constants such that

$$((b-a) \sup_{t \in J} \int_a^b \left(\frac{\partial^i K_0(t,s)}{\partial t^i}\right)^2 ds)^{1/2} \leq M_i, \quad i=0,\ldots,n-1 \tag{4.2.5}$$

and $K_0(t,s)$ is the function corresponding to L_0; r is a non-negative number such that

$$|| \tau\bar{x} - N\bar{x}| \leq r\sqrt{b-a} \ . \tag{4.2.6}$$

Then the MPBVP (4.1.2) has a unique solution $x = \hat{x}$ such that $\hat{x}(t) \in D_\delta$ and this is an isolated solution. Furthermore,

$$|\hat{x}^{(i)}(t) - \bar{x}^{(i)}(t)| \leq \frac{\tilde{M}r}{1-\rho}, \quad i=0,\ldots,n-1. \tag{4.2.7}$$

Proof: As usual, we define a pseudo-norm $\mu(.)$ on $\tilde{H}^{n-1}(J)$ as follows:

$$\mu(x) = \max\left(\max_{i=0,\ldots,n-2} \sup_{t\in J} |x^{(i)}(t)|, \ \text{ess.} \sup_{t \in J} |x^{(n-1)}(t)|\right)$$

for $x \in \tilde{H}^{n-1}(J)$.

Since $\bar{x} \in D(N) \cap D(L)$ is an approximate solution of (4.1.2), we have

$$\tau\bar{x} = N\bar{x} + \eta, \ B_j(x) = 0, \ j=1,2,\ldots,k \tag{4.2.8}$$

where η is the residue function belonging to S.

We define

$$\tilde{H}_{\delta}^{n-1}(J) = \{x \in \tilde{H}^{n-1}(J): |x^{(i)}(t)-\bar{x}^{(i)}(t)| \leq \delta,\ i=0,\ldots,n-1 \text{ for all } t \in J \text{ for which } x^{(n-1)}(t) \text{ exists}\}.$$

We notice that if $x \in \tilde{H}_{\delta}^{n-1}(J)$, then $(x(t),x^{(1)}(t),\ldots,x^{(n-1)}(t)) \in D_{\delta}$. Therefore, by (4.2.4(ii)), $\tilde{H}_{\delta}^{n-1}(J) \subset D(N)$. Let us rewrite the equation (4.2.8) as follows:

$$\tau\bar{x} = N\bar{x} + \eta - \tau_0\bar{x} + \tau_0\bar{x},\ B_j(x)=0,\ j=1,2,\ldots,k.$$

That is

$$(\tau-\tau_0)\bar{x} = N\bar{x} + \eta - \tau_0\bar{x},$$

$$B_j(x) = 0,\ j=1,2,\ldots,k.$$

Hence, as in (1.1.3), we have

$$L_0\bar{x} = N\bar{x} + \eta - \tau_0\bar{x}.$$

Therefore, by (4.2.4(i)) and lemma 4.1, we have

$$\bar{x}(t) = \int_a^b K_0(t,s)[(N\bar{x} + \eta - \tau_0\bar{x})(s)]ds,\ t \in J. \quad (4.2.9)$$

Let us consider the iterative process

$$x_{m+1}(t) = \int_a^b K_0(t,s)[(Nx_m - \tau_0 x_m)(s)]ds,\ m=0,1,\ldots, \quad (4.2.10)$$

where $x_0 = \bar{x}$.

We note that the sequence $\{x_m\} \subset D(L_0) \subset H^n(J)$. Firstly, we shall prove that the iterative process can be continued

indefinitely in $\tilde{H}_{\delta}^{n-1}(J)$, and that

$$\mu(x_{m+1} - x_m) \leq \rho^m \mu(x_1 - x_o), \tag{4.2.11}$$

$$\mu(x_{m+1} - x_o) \leq \delta, \ (m=0,1,\ldots). \tag{4.2.12}$$

For $m=0$, (4.2.11) is evident. Since

$$x_1(t) - x_o(t) = -\int_a^b K_o(t,s)\ \eta(s)ds,$$

we have

$$\mu(x_1 - x_o) = \max_{i=0,\ldots,n-1} \left(\sup_{t \in J} \left| \int_a^b \frac{\partial^i K_o(t,s)}{\partial t^i} \eta(s)ds \right| \right)$$

$$\leq \max_{i=0,\ldots,n-1} \left(\left(\sup_{t \in J} \int_a^b \left(\frac{\partial^i K_o(t,s)}{\partial t^i} \right)^2 ds \right)^{1/2} ||\eta|| \right)$$

(by Schwartz inequality)

$$\leq \max_{i=0,\ldots,n-1} \left(\left(\sup_{t \in J} \int_a^b \left(\frac{\partial^i K_o(t,s)}{\partial t^i} \right)^2 ds \right)^{1/2} r\sqrt{b-a} \right)$$

(by (4.2.6))

$$\leq \max_{i=0,\ldots,n-1} M_i\ r \qquad \text{(by (4.2.5))}$$

$$= \tilde{M}r \tag{4.2.13}$$

$$\leq (1-\rho)\delta < \delta \qquad \text{(by (4.2.4(iv)))}.$$

This proves (4.2.12) for $m = 0$.

Let us assume that (4.2.11) and (4.2.12) hold upto m-1. Then by (4.2.10), we have

$$x_{m+1}(t) - x_m(t) = \int_a^b K_o(t,s) \left[(Nx_m - Nx_{m-1} - \tau_o(x_m - x_{m-1}))(s) \right] ds,$$

and hence

$$\mu(x_{m+1}-x_m) \leq \max_{i=0,\ldots,n-1}((\sup_{t\in J}\int_a^b(\frac{\partial^i K_o(t,s)}{\partial t^i})^2 ds)^{1/2}$$

$$||Nx_m-Nx_{m-1}-\tau_o(x_m-x_{m-1})|| \text{ (by Schwartz inequality)}$$

$$\leq \max_{i=0,\ldots,n-1}(((b-a)\sup_{t\in J}\int_a^b(\frac{\partial^i K_o(t,s)}{\partial t^i})^2 ds)^{1/2}$$

$$\sup_{s\in J}|(Nx_m-x_{m-1}-\tau_o(x_m-x_{m-1}))(s)|). \quad (4.2.14)$$

But,

$$|(Nx_m-Nx_{m-1}-\tau_o(x_m-x_{m-1}))(s)|$$

$$= |X(s,x_m(s),\ldots,x_m^{(n-1)}(s))-X(s,x_{m-1}(s),\ldots,x_{m-1}^{(n-1)}(s))$$

$$-(\tau_o(x_m-x_{m-1}))(s)|$$

$$= |\int_0^1(\sum_{i=0}^{n-1}\left[\frac{\partial X(s,x_{m-1}(s)+\nu(x_m(s)-x_{m-1}(s)),\ldots,x_{m-1}^{n-1}(s)+\nu(x_m^{(n-1)}(s)-x_{m-1}^{(n-1)}(s))}{\partial x_i}\right.$$

$$\left.(x_m^{(i)}(s)-x_{m-1}^{(i)}(s))\right]-(\tau_o(x_m-x_{m-1}))(s))d\nu|. \quad (4.2.15)$$

We notice that

$$|x_{m-1}^{(i)}(s)+\nu(x_m^{(i)}(s)-x_{m-1}^{(i)}(s))-x_o^{(i)}(s)|$$

$$\leq (1-\nu)|x_{m-1}^{(i)}(s)-x_o^{(i)}(s)|+\nu|x_m^{(i)}(s)-x_o^{(i)}(s)|$$

$$\leq (1-\nu)\delta+\nu\delta = \delta, \quad i=0,\ldots,n-1.$$

The equation (4.2.15) can be rewritten as follows :

$$|X(s,x_m(s),\ldots,x_m^{(n-1)}(s))-X(s,x_{m-1}(s),\ldots,x_{m-1}^{(n-1)}(s))$$

$$-(\tau_0(x_m-x_{m-1}))(s)|$$

$$= \left|\int_0^1 \left(\sum_{i=0}^{n-1} \left[\frac{\partial X(s,x_{m-1}(s)+\nu(x_m(s)-x_{m-1}(s)),\ldots,x_{m-1}^{(n-1)}(s)+\nu(x_m^{(n-1)}(s)-x_{m-1}^{(n-1)}(s)))}{\partial x_i} - q_i(s)\right](x_m^{(i)}(s)-x_{m-1}^{(i)}(s)\right)d\nu\right|$$

(by (4.2.1))

$$\leq \sum_{i=0}^{n-1} \left|\frac{\partial X(s,x_{m-1}(s)+\nu(x_m(s)-x_{m-1}(s)),\ldots,x_{m-1}^{(n-1)}(s)+\nu(x_m^{(n-1)}(s)-x_{m-1}^{(n-1)}(s)))}{\partial x_i} - q_i(s)\right| |x_m^{(i)}(s)-x_{m-1}^{(i)}(s)|$$

$$\leq \mu(x_m-x_{m-1})\sum_{i=0}^{n-1} \left|\frac{\partial X(s,x_{m-1}(s)+\nu(x_m(s)-x_{m-1}(s)),\ldots,x_{m-1}^{(n-1)}(s)+\nu(x_m^{(n-1)}(s)-x_{m-1}^{(n-1)}(s)))}{\partial x_i} - q_i(s)\right|$$

$$\leq \mu(x_m-x_{m-1})\frac{\rho}{\tilde{M}} \quad \text{(by (4.2.4(iii)))}. \tag{4.2.16}$$

Thus from (4.2.14), (4.2.15) and (4.2.16), we get

$$\mu(x_{m+1}-x_m) \leq \tilde{M}\frac{\rho}{\tilde{M}}\ \mu(x_m-x_{m-1}) = \rho\mu(x_m-x_{m-1}).$$

That is

$$\mu(x_{m+1}-x_m) \leq \rho\mu(x_m-x_{m-1}). \tag{4.2.17}$$

But $\qquad \mu(x_m - x_{m-1}) \leq \rho^{m-1}\, \mu(x_1 - x_o).$

Therefore, from (4.2.17), we get

$$\mu(x_{m+1} - x_m) \leq \rho^m\, \mu(x_1 - x_o).$$

This proves (4.2.11) for m.

Also,

$$\begin{aligned}
\mu(x_{m+1} - x_o) &\leq \mu(x_{m+1} - x_m) + \mu(x_m - x_{m-1}) + \dots + \mu(x_1 - x_o) \\
&\leq (\rho^m + \rho^{m-1} + \dots + \rho + 1)\, \mu(x_1 - x_o) \\
&\leq \frac{1}{1-\rho}\, \mu(x_1 - x_o) \\
&\leq \frac{\tilde{M} r}{1-\rho} \qquad \text{(by (4.2.13))} \\
&\leq \delta \qquad \text{(by (4.2.4(iv)))}.
\end{aligned}$$

This proves (4.2.12) for m.

By (4.2.11) and (4.2.12) it is evident that the iterative process (4.2.10) can be continued indefinitely in $\tilde{H}_\delta^{n-1}(J)$. Thus we have a sequence $\{x_m\} \subset D(L_o)$, $\{x_m\} \subset \tilde{H}_\delta^{n-1}(J)$ and this sequence $\{x_m\}$ together with all its derivatives upto order (n-1) are uniformly convergent by (4.2.11), since $\rho < 1$. Therefore, there exists a function $\hat{x} \in C^{n-1}(J)$ such that

$$\hat{x}(t) = \lim_{t \to \infty} x_m(t), \dots, \hat{x}^{(n-1)}(t) = \lim_{t \to \infty} x_m^{(n-1)}(t).$$

Moreover, $\hat{x} \in \tilde{H}_\delta^{n-1}(J)$ by (4.2.12).

For this limit function $\hat{x}$, (4.2.10) yields that

$$\hat{x}(t)-\int_a^b K_0(t,s)\left[X(s,\hat{x}(s),\ldots,\hat{x}^{n-1}(s)) - (\tau_0\hat{x})(s)\right] ds$$

$$= \hat{x}(t)-x_{m+1}(t)+\int_a^b K_0(t,s)\left[X(s,x_m(s),\ldots,x_{m-1}^{(n-1)}(s))\right.$$

$$- X(s,\hat{x}(s),\ldots,\hat{x}^{(n-1)}(s))$$

$$\left.- (\tau_0(x_m - \hat{x}))(s)\right] ds.$$

Then analogous to (4.2.17), we get

$$\mu(\int_a^b K_0(\cdot,s)\left[X(s,x_m(s),\ldots,x_m^{(n-1)}(s))-X(s,\hat{x}(s),\ldots,\hat{x}^{(n-1)}(s))\right.$$

$$\left.- (\tau_0(\hat{x}_m - x))(s)\right] ds)$$

$$\leq \rho\mu(x_m - \hat{x}).$$

Thus we have

$$\mu(\hat{x} - \int_a^b K_0(\cdot,s)\left[(N\hat{x} - \tau_0\hat{x})(s)\right] ds)$$

$$\leq \mu(\hat{x} - x_{m+1}) + \rho\mu(x_m - \hat{x}).$$

Letting $m \to \infty$, we see that the right hand side of the above inequality tends to zero.

Hence,

$$\hat{x} = \int_a^b K_0(\cdot,s)\left[(N\hat{x})(s) - (\tau_0\hat{x})(s)\right] ds. \tag{4.2.18}$$

Therefore,

$$L_0\hat{x} = N\hat{x} - \tau_0\hat{x}.$$

That is

$$(\tau - \tau_0)\hat{x} = N\hat{x} - \tau_0\hat{x},\ B_j(\hat{x}) = 0,\ j = 1,2,\dots,k.$$

Thus

$$\tau\hat{x} = N\hat{x},\ B_j(\hat{x}) = 0, j = 1,2,\dots,k.$$

Hence,

$$L\hat{x} = N\hat{x}.$$

Moreover, since $\mu(x_{m+1} - x_0) \leq \frac{\tilde{M}r}{1-\rho}$, letting $m \to \infty$ we get

$$\mu(\hat{x} - \bar{x}) \leq \frac{\tilde{M}r}{1-\rho} .$$

This proves (4.2.7).

We now prove the uniqueness of the solution.

Let $\hat{\hat{x}}$ be another solution of (4.1.2) lying in $\tilde{H}_\delta^{n-1}(J)$. Then

$$L\hat{\hat{x}} = N\hat{\hat{x}}.$$

This can be rewritten as

$$\tau\hat{\hat{x}} - \tau_0\hat{\hat{x}} = N\hat{\hat{x}} - \tau_0\hat{\hat{x}},\ B_j(\hat{\hat{x}}) = 0,\ j = 1,2,\dots,k.$$

Therefore, $\hat{\hat{x}}$ can be expressed as follows :

$$\hat{\hat{x}}(t) = \int_a^b K_0(t,s) \left[(N\hat{\hat{x}} - \tau_0\hat{\hat{x}})(s) \right] ds .$$

Therefore,

$$\hat{x}(t) - \hat{\hat{x}}(t) = \int_a^b K_0(t,s) \left[(N\hat{x} - N\hat{\hat{x}} - \tau_0(\hat{x} - \hat{\hat{x}}))(s) \right] ds$$

(see(4.2.18)).

Then analogous to (4.2.17), we get

$$\mu(\hat{x} - \hat{\hat{x}}) \leq \rho\mu(\hat{x} - \hat{\hat{x}})$$

which implies that $\mu(\hat{x}-\hat{\hat{x}}) = 0$ because $\rho < 1$. Thus, $\hat{x} = \hat{\hat{x}}$ and hence the uniqueness is proved.

Finally, we establish the isolatedness of the solution $x = \hat{x}$. Put

$$\tau_1 y = \sum_{i=0}^{n-1} \frac{\partial X(t,\hat{x}(t),\ldots,\hat{x}^{(n-1)}(t))}{\partial x_i} \frac{d^i y}{dt^i}.$$

Let us consider the linear homogeneous MPBVP

$$\tau y - \tau_1 y = 0, \qquad (4.2.19)$$

$$B_j(y) = 0, \; j = 1,2,\ldots,k. \qquad (4.2.20)$$

By (4.2.4(iii)), we have

$$\sum_{i=0}^{n-1} \left| \frac{\partial X(t,\hat{x}(t),\ldots,\hat{x}^{(n-1)}(t))}{\partial x_i} - q_i(t) \right| \leq \rho/\tilde{M}. \qquad (4.2.21)$$

Any solution of (4.2.19) - (4.2.20) satisfies

$$\tau y - \tau_o y = (\tau_1 - \tau_o)y, \; B_j(y) = 0, j = 1,\ldots,k.$$

That is

$$L_o y = (\tau_1 - \tau_o)y.$$

Consequently y can be expressed as follows :

$$y(t) = \int_a^b K_o(t,s) \left[(\tau_1 y - \tau_o y)(s) \right] ds$$

$$= \int_a^b K_o(t,s) \left[\sum_{i=0}^{n-1} \left(\frac{\partial X(s,\hat{x}(s),\ldots,\hat{x}^{(n-1)}(s))}{\partial x_i} - q_i(s) \right) \frac{d^i y(s)}{ds^i} \right] ds.$$

Therefore,

$$\mu(y) \le \tilde{M} \sup_{s \in J} \sum_{i=0}^{n-1} \left| \frac{\partial X(s,\hat{x}(s),\ldots,\hat{x}^{(n-1)}(s))}{\partial x_i} - q_i(s) \right| \mu(y)$$

(analogous to (4.2.14))

$$\le \tilde{M} \frac{\rho}{\tilde{M}} \mu(y) \qquad \text{(by (4.2.21))}$$

$$= \rho\mu(y)$$

which implies that $\mu(y) = 0$, since $\rho < 1$. This shows that $y = 0$ and hence there is no nontrivial solution of (4.2.19) – (4.2.20). Thus $x = \hat{x}$ is an isolated solution of (4.1.2). This completes the proof of the theorem.

CHAPTER 5

AN EXISTENTIAL ANALYSIS FOR A NONLINEAR DIFFERENTIAL EQUATION WITH NONLINEAR MULTI-POINT BOUNDARY CONDITIONS

5.0. OUTLINE OF THE CHAPTER

In this chapter we develop an existential analysis for the following MPBVP with nonlinear boundary conditions:

$$\tau x = Nx, \tag{5.0.1}$$

$$f_j(x) = 0, \; j=1,2,\ldots,k, k \leq n \tag{5.0.2}$$

over the interval J where τ is the operator defined by (1.1.1), and the function N and the real-valued functions f_js are defined subsequently.

The following example is worked out in detail to illustrate the method:

$$\begin{aligned} &x^{(2)} + x = \frac{x^3}{2}, \\ &\frac{1}{8}(x^{(1)}(0) + x^{(1)}(\pi))^3 - x^{(1)}(2\pi) = 0, \qquad (5.0.3) \\ &\frac{1}{8}(x(0) + x(2\pi))^3 + x(\pi) = 0 \end{aligned}$$

over the interval $[0,2\pi]$.

5.1. NOTATIONS AND ASSUMPTIONS

We assume the following:

(i) We take τ to be the operator defined by (1.1.1). It is

assumed that the coefficient functions in τ belong to $C^{\infty}(J)$ and the leading coefficient $p_n(t) \neq 0$ on J.

(ii) Let $X(t,x_0,\ldots,x_{n-1})$ be a nonlinear real-valued function defined for $t \in J$ and $|x_i| \leq R_i$, $i=0,\ldots,n-1$ where each $R_i > 0$.

(iii) $X(\cdot, x_0,\ldots,x_{n-1}) \in S$ for each fixed $(x_0,\ldots,x_{n-1})$ such that $|x_i| \leq R_i$.

(iv) The exists a real number $k_0 \geq 0$ such that for $|x_i| \leq R_i$ and $|y_i| \leq R_i$ we have

$$|X(t,x_0,\ldots,x_{n-1})-X(t,y_0,\ldots,y_{n-1})| \leq k_0\left(\sum_{i=0}^{n-1} |x_i-y_i|\right), t \in J. \tag{5.1.1}$$

We define the operator N as follows:

$$D(N) = \{x \in \tilde{H}^{n-1}(J): \sup_{t \in J} |x^{(i)}(t)| \leq R_i,\ i=0,\ldots,n-1,\ \operatorname{ess.}\sup_{t \in J} |x^{(n-1)}(t)| \leq R_{n-1}\}, \tag{5.1.2}$$

$$(Nx)(t) = X(t,x(t),x^{(1)}(t),\ldots,x^{(n-1)}(t)) \text{ for all } t \in J \text{ for which } |x^{(n-1)}(t)| \leq R_{n-1}.$$

From the assumption (iii) and relation (5.1.1), it is clear that $Nx \in S$ for $x \in D(N)$.

For $j \in \{1,2,\ldots,k\}$, $k \leq n$, we denoted by

$$f_j(x) = g_j(x(a),x^{(1)}(a),\ldots,x^{(n-1)}(a);\ x(a_1),\ x^{(1)}(a_1),\ldots,x^{(n-1)}(a_1);,\ldots;\ x(b),x^{(1)}(b),\ldots,x^{(n-1)}(b)) \tag{5.1.3}$$

for all $x \in C^{n-1}(J)$ where $a \leq a_1 \leq a_2 \leq \cdots \leq a_{h-1} \leq b$ and $g_j(\cdot,\ldots,\cdot)$ is a nonlinear real-valued function of the variables.

We assume the following on f_js:

(v) There exist real constants $\ell_j \geq 0$, $j = 1,2,\ldots,k$ such that for $x,y \in C^{n-1}(J)$, we have

$$|f_j(x)-f_j(y)| \leq \ell_j\Big(\max_{i=0,\ldots,n-1} \sup_{t\in J} |x^{(i)}(t)-y^{(i)}(t)|\Big). \tag{5.1.4}$$

We solve the following nonlinear MPBVP:

$$\tau x = Nx, \tag{5.1.5}$$

$$f_j(x)= 0,\ j=1,\ldots,k,\ k \leq n \tag{5.1.6}$$

over the interval J where τ is given by (1.1.1), N is given by (5.1.2) and f_js are given by (5.1.3). Throughout the chapter τ, N, f_js stands for the quantities defined above.

5.2. OPERATOR $T_1(\tau)$ AND IT'S PROPERTIES

We consider the operator $T_1(\tau)$ defined by (1.1.4). From section 1.1, we recollect the following facts about $T_1(\tau)$:

(i) $D(T_1(\tau))$ is dense in S.

(ii) $T_1(\tau)$ is a closed linear operator.

(iii) $T_1(\tau)^* = T_o(\tau^*)$.

(iv) $\dim N(T_1(\tau)) = n$.

As we did in section 1.1, we choose functions $\phi_1,\ldots,\phi_n \in C^\infty(J)$ to form an orthonormal basis for $N(T_1(\tau))$. Throughout this chapter, we take $G(\cdot,\cdot)$ to be the function defined by (1.3.2). For $y \in S$, we consider the function u defined by (1.3.3). We now assert that $R(T_1(\tau)) = S$. Indeed, suppose $y \in S$. Consider the function u defined by (1.3.3) corresponding to y. Then by Lemma 1.2, we have $u \in H^n(J)$ and $T_1(\tau)u = \tau u = y$. Thus $R(T_1(\tau)) = S$. Moreover, since $S = R(T_1(\tau)) \oplus N(T_0(\tau^*))$, we have $N(T_0(\tau^*)) = \phi$. Thus dim $N(T_0(\tau^*)) = 0$.

5.3. OPERATORS F, P_m, Q_m AND CERTAIN RELATIONS INVOLVING $T_1(\tau)$, F, P_m, Q_m

We note that the operator $T_1(\tau)|H^n(J) \cap N(T_1(\tau))^\perp$ is a closed one-to-one operator having the same range as $T_1(\tau)$. Let F denote the inverse of this operator:

$$F = (T_1(\tau)|H^n(J) \cap N(T_1(\tau))^\perp)^{-1}. \tag{5.3.1}$$

By the Closed Graph Theorem, F is a one-to-one continuous linear operator. Clearly, $D(F) = R(T_1(\tau)) = S$ and $R(F) = D(T_1(\tau)) \cap N(T_1(\tau))^\perp = H^n(J) \cap N(T_1(\tau))^\perp$. Moreover,

$$T_1(\tau)Fy = y \quad \text{for all} \quad y \in S, \tag{5.3.2}$$

and

$$FT_1(\tau)x = x - \sum_{i=1}^{n} (x,\phi_i)\phi_i \quad \text{for all} \quad x \in H^n(J). \tag{5.3.3}$$

Thus F is a continuous right inverse of $T_1(\tau)$.

Now, let us choose elements $\omega_1, \omega_2, \ldots, \omega_m, \ldots \in D(T_o(\tau^*)) = H_o^n(J)$ to form a complete orthonormal set in S. The existence of such a sequence in $H_o^n(J)$ is not difficult to show. For example, consider the following selfadjoint eigenvalue problem:

$$
\begin{aligned}
x^{(2n)} &= \lambda x, \\
x(a) &= x^{(1)}(a) = \ldots = x^{(n-1)}(a) = 0, \\
x(b) &= x^{(1)}(b) = \ldots = x^{(n-1)}(b) = 0.
\end{aligned}
$$

Clearly, zero is not an eigenvalue of the above problem. We also know that the above problem has infinitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m, \ldots$. and the corresponding eigenfunctions $\omega_1, \omega_2, \ldots, \omega_m, \ldots$ form a complete orthonormal set in S. Obviously, the functions $\omega_1, \omega_2, \ldots, \omega_m, \ldots \in H_o^n(J)$.

Take $m \geq 1$. Let S_o be the $m+n$-dimensional space spanned by the functions $\phi_1, \phi_2, \ldots, \phi_n$, and $F\omega_1, F\omega_2, \ldots, F\omega_m$. That is

$$S_o = \langle \phi_1, \phi_2, \ldots, \phi_n, F\omega_1, F\omega_2, \ldots, F\omega_m \rangle . \tag{5.3.4}$$

The sequences of projections $\{P_m\}$ and $\{Q_m\}$ on S is defined as follows:

$$P_m x = \sum_{i=1}^{m} (x, \omega_i)\omega_i \quad \text{for all } x \in S, \tag{5.3.5}$$

and

$$Q_m x = \sum_{i=1}^{n} (x, \phi_i)\phi_i + \sum_{i=1}^{m} (x, T_o(\tau^*)\omega_i)F\omega_i \quad \text{for all } x \in S. \tag{5.3.6}$$

The operators P_m and Q_m have the following properties:

(i) $R(P_m) = \langle \omega_1, \ldots, \omega_m \rangle$.

(ii) $R(Q_m) = S_o \subset H^n(J)$.

(iii) $P_m^2 = P_m$ and $Q_m^2 = Q_m$.

We now state the following theorem.

THEOREM 5.1. The following relations are valid:

(i) $F(I-P_m)\, T_1(\tau)x = (I-Q_m)x$ for all $x \in H^n(J)=D(T_1(\tau))$.

(ii) $T_1(\tau)F(I-P_m)x = (I-P_m)x$ for all $x \in S$.

(iii) $T_1(\tau)Q_m x = P_m T_1(\tau)x$ for all $x \in H^n(J)$.

(iv) $Q_m H(I-P_m)x = 0$ for all $x \in S$.

Proof of the above theorem is similar to the proof of theorem 1.1 and can be verified easily.

5.4. CERTAIN INTEGRAL REPRESENTATION FOR F AND $F(I-P_m)$

The following theorem gives an integral representation for F.

THEOREM 5.2. Let $y \in S$. Then Fy has the representation given by

$$(Fy)(t) = \sum_{i=1}^{n} (\psi_i, y)\phi_i(t) + \int_a^t G(t,s)y(s)ds, \quad t \in J \tag{5.4.1}$$

where

$$\psi_i(t) = -\int_t^b \phi_i(s)G(s,t)ds, \quad t \in J. \tag{5.4.2}$$

Proof: Take $y \in S$. Let $x = Fy$, and let $u(t) = \int_a^b G(t,s)y(s)ds$, $t \in J$. Using basic properties of F together with Lemma 1.1, we get $x, u \in H^n(J)$ and $T_1(\tau)(x-u) = \tau(x-u) = 0$. Thus $x-u \in N(T_1(\tau))$. Hence there exist real constants $c_1, c_2, \ldots, c_n$ such that

$$x = \sum_{i=1}^{n} c_i \phi_i + u. \tag{5.4.3}$$

Since $x \in N(T_1(\tau))^{\perp}$, we have $(x,\phi_i) = 0$, $i=1,\ldots,n$. Thus from (5.4.3), we get

$$c_i\delta_i = -(u,\phi_i),\ i=1,2,\ldots,n. \tag{5.4.4}$$

On the other hand,

$$\begin{aligned}
(u,\phi_i) &= \int_a^b \left(\int_a^t G(t,s)y(s)ds\right)\phi_i(t)dt \\
&= \int_a^b \int_a^t G(t,s)y(s)\phi_i(t)ds\ dt \\
&= \int_a^b \int_s^b G(t,s)y(s)\phi_i(t)dt\ ds \quad \text{(by Fubini's theorem)} \\
&= \int_a^b y(s) \left(\int_s^b G(t,s)\phi_i(t)dt\right)ds \\
&= -\int_a^b y(s)\ \psi_i(s)ds \qquad \text{(by (5.4.2))} \\
&= -(\psi_i, y).
\end{aligned} \tag{5.4.5}$$

Thus, from (5.4.3), (5.4.4) and (5.4.5), we get the required representation. This completes the proof of the theorem.

Let $K(\cdot,\cdot)$ be the function defined on the square $J \times J$ by

$$K(t,s) = \begin{cases} \sum_{i=1}^{n} \phi_i(t)\, \psi_i(s) + G(t,s) & \text{for } a \leq s \leq t \leq b, \\ \sum_{i=1}^{n} \phi_i(t)\, \psi_i(s) & \text{for } a \leq t \leq s \leq b . \end{cases} \tag{5.4.6}$$

We note that the function $K(\cdot,s)$ is continuous together with all its derivatives upto order (n-2) on J, while $\frac{\partial^{n-1} K(\cdot,s)}{\partial t^{n-1}}$ is discontinuous at $t = s$ with the jump given by

$$\frac{\partial^{n-1} K(s+0,s)}{\partial t^{n-1}} - \frac{\partial^{n-1} K(s-0,s)}{\partial t^{n-1}} = \frac{1}{p_n(s)} .$$

We now state a corollary of theorem 5.2.

COROLLARY. The right inverse operator F has an integral representation given by

$$(Fy)(t) = \int_a^b K(t,s)y(s), \quad t \in J \tag{5.4.7}$$

for all $y \in S$.

Proof of the corollary follows from (5.4.6) and theorem 5.2.

Let $K_m(\cdot,\cdot)$ be the function defined on $J \times J$ by

$$K_m(t,s) = K(t,s) - \sum_{i=1}^{m} \left(\int_a^b K(t,\xi)\, \omega_i(\xi)d\xi \right) \omega_i(s), \quad a \leq t,s \leq b. \tag{5.4.8}$$

We notice that $\frac{\partial^i K_m(\cdot,\cdot)}{\partial t^i}$, $i = 0,\ldots,n$ are square-integrable on $J \times J$, while the functions $\int_a^b \left(\frac{\partial^i K_m(\cdot,s)}{\partial t^i}\right)^2 ds$, $i=0,1,\ldots,n-1$ are continuous on J.

The following theorem gives an integral representation for $F(I-P_m)$.

THEOREM 5.3. The linear operator $F(I-P_m)$ has an integral representation given by

$$(F(I-P_m)x)(t) = \int_a^b K_m(t,s)x(s)ds, \quad t \in J \tag{5.4.9}$$

for all $x \in S$.

Proof of the above theorem is analogous to the proof of theorem 1.3 and can be verified easily.

5.5. SPACE Y AND OPERATORS $\bar{I}$, L, H, AND CERTAIN INTEGRAL REPRESENTATION FOR H

Let us consider the Banach space $S \times R^k$ with the product topology. We denote by

$$Y = S \times R^k.$$

For cl~~e~~arity, the identity operator on Y is denoted by $\bar{I}$ and the norm on Y is denoted by $||\cdot||_Y$.

We define the operator $L:D(L) \to Y$ as follows:

$$\begin{aligned} D(L) &= D(T_1(\tau)) = H^n(J) \\ Lx &= (T_1(\tau)x,0) \end{aligned} \tag{5.5.1}$$

where '0' is the zero element of R^k.

Since $T_1(\tau)$ is a closed linear operator and $R(T_1(\tau)) = S$, we have

(i) $D(L) = H^n(J)$ is dense in S.

(ii) $R(L) = S \times \{0\}$ where '0' is the zero element of R^k.

(iii) L is a closed linear operator.

(iv) $N(L) = N(T_1(\tau))$.

Moreover, $L|H^n(J) \cap N(L)^{\perp}$ is a one-to-one closed linear operator having the same range as L. Let H denote the inverse of this operator:

$$H = (L|D(L) \cap N(L)^{\perp})^{-1}. \tag{5.5.2}$$

We notice that $R(L)$ is closed in Y. Therefore, by the closed graph theorem, H is a one-to-one continuous linear operator. Clearly, $D(H) = R(L) = S \times \{0\}$ and $R(H) = D(L) \cap N(L)^{\perp} = H^n(J) \cap N(T_1(\tau))^{\perp}$. Moreover,

$$LHy = y \quad \text{for all} \quad y \in S \times \{0\} = R(L). \tag{5.5.3}$$

and

$$HLx = x - \sum_{i=1}^{n} (x, \phi_i)\phi_i \text{ for all } x \in H^n(J) = D(L). \tag{5.5.4}$$

Thus H is the right inverse of L.

We notice that if $y = (z,0) \in R(L) = S \times \{0\}$, then the function u given by

$$u(t) = \int_a^t G(t,s)z(s)ds, \; t \in J \tag{5.5.5}$$

belong to $H^n(J)$ and $Lu = (T_1(\tau)u,0) = (z,0) = y$.

We now prove the following theorem.

THEOREM 5.4. Let $y = (z,0) \in R(L)$. Then Hy has the representation given by

$$(Hy)(t) = \sum_{i=1}^{n} (\psi_i, z)\, \phi_i(t) + \int_a^t G(t,s)z(s)ds, \quad t \in J \tag{5.5.6}$$

where ψ_is are given by (5.4.2).

Proof: Take $y = (z,0) \in R(L)$. Let $x = Hy$ and

$u(t) = \int_a^t G(t,s)z(s)ds$, $t \in J$. Then we get $L(x-u) = 0$. Hence $x-u \in N(L) = N(T_1(\tau))$. Therefore there exist real constants $c_1, c_2, \ldots, c_n$ such that

$$x = u + \sum_{i=1}^{n} c_i \phi_i \,.$$

Then proceeding similarly as in the proof of theorem 5.2, we get the required representation. Thus the proof is completed.

REMARK. From the representation (5.5.6), we have

$$Hy = Fz \tag{5.5.7}$$

where $y = (z,0) \in R(L)$.

5.6. PROJECTION $\bar{P}_m$ AND CERTAIN RELATIONS INVOLVING $L,H,\bar{P}_m,Q_m$ AND CERTAIN INTEGRAL REPRESENTATION FOR $H(\bar{I}-\bar{P}_m)$

For $m \geq 1$, let us define the sequence of projections $\{\bar{P}_m\}$ on Y as follows:

$$\bar{P}_m y = (P_m z, \alpha) \quad \text{for all} \quad y = (z,\alpha) \in Y. \tag{5.6.1}$$

We notice the following properties:

(i) $\bar{P}_m$ is a continuous linear operator defined on all of Y.

(ii) $R(\bar{P}_m) = \langle \omega_1, \omega_2, \dots, \omega_m \rangle \times R^k$.

(iii) The range of $\bar{I}-\bar{P}_m$ is a subset of $R(L)$.

(iv) $H(\bar{I}-\bar{P}_m)$ is a continuous linear operator defined on all of Y.

We now prove the following theorem.

THEOREM 5.5. The following relations are valid:

(i) $H(\bar{I}-\bar{P}_m)Lx = (I-Q_m)x$ for all $x \in D(L)$.

(ii) $LH(\bar{I}-\bar{P}_m)y = (\bar{I}-\bar{P}_m)y$ for all $y \in Y$.

(iii) $LQ_m x = \bar{P}_m Lx$ for all $x \in D(L)$.

(iv) $Q_m H(\bar{I}-\bar{P}_m)y = 0$ for all $y \in Y$.

Proof: (i) Take $x \in D(L)$. Then

$$Lx = (T_1(\tau)x, 0).$$

Therefore,

$$(\bar{I}-\bar{P}_m)Lx = ((I-P_m)T_1(\tau)x, 0)$$

Hence

$$\begin{aligned} H(\bar{I}-\bar{P}_m)Lx &= H((I-P_m)\,T_1(\tau)x, 0) \\ &= F(I-P_m)\,T_1(\tau)x \qquad \text{(by (5.5.7))} \\ &= (I-Q_m)x \qquad \text{(by theorem 5.1(i))}. \end{aligned}$$

(ii) Take $y \in Y$. Since $(\bar{I}-\bar{P}_m)y \in R(L)$ and H is the right inverse of L, we have

$$LH(\bar{I}-\bar{P}_m)y = (\bar{I}-\bar{P}_m)y.$$

(iii) Take $x \in D(L)$. Then

$$\begin{aligned} LQ_m x &= (T_1(\tau)Q_m x, 0) \\ &= (P_m T_1(\tau)x, 0) \qquad \text{(by theorem 5.1(iii))} \\ &= \bar{P}_m (T_1(\tau)x, 0) \\ &= \bar{P}_m Lx. \end{aligned}$$

(iv) Take $y = (z,\alpha) \in Y$. Then

$$\begin{aligned} Q_m H(\bar{I}-\bar{P}_m)y &= Q_m H((I-P_m)z, 0) \\ &= Q_m F(I-P_m)z \qquad \text{(by (5.5.7))} \\ &= 0 \qquad \text{(by theorem 5.1(iv)).} \end{aligned}$$

Thus the proof of theorem is completed.

The next theorem gives an integral representation for $H(\bar{I}-\bar{P}_m)$.

THEOREM 5.6. The linear operator $H(\bar{I}-\bar{P}_m)$ has an integral representation given by

$$(H(\bar{I}-\bar{P}_m)y)(t) = \int_a^b K_m(t,s)z(s)ds$$

for all $y = (z,\alpha) \in Y$ where $K_m(\cdot,\cdot)$ is the function defined in (5.4.8).

Proof: Take $y = (z,\alpha) \in Y$. Then

$$(\bar{I}-\bar{P}_m)y = ((I-P_m)z,0).$$

Then by (5.5.7), we have

$$H(\bar{I}-\bar{P}_m)y = H((I-P_m)z,0)$$

$$= F(I-P_m)z. \qquad (5.6.2)$$

Now, the theorem 5.3 yields the required representation. Thus the proof of the theorem is completed.

5.7. EXISTENTIAL ANALYSIS

In this section we follow the notations of earlier sections of the chapter and develop an existential theory for the nonlinear MPBVP (5.1.5)-(5.1.6). Towards this end, we denote by

$$f = \nu(f_1,f_2,\ldots,f_k) \qquad (5.7.1)$$

where $f_1,\ldots,f_k$, $k \leq n$, are the functions defined by (5.1.3) and ν is a sufficiently small positive real number.

We solve the equation

$$Lx = (Nx,\ f(x)) \qquad (5.7.2)$$

where L is defined by (5.5.1) and N is defined by (5.1.2). If the equation (5.7.2) has a solution, then the equation

$$(T_1(\tau)\ x,0) = (Nx,\ f(x))$$

has a solution which readily implies that the MPBVP (5.1.5)-(5.1.6) has a solution. For simplicity, we denote by

$$\bar{N}x = (Nx,\ f(x)) \tag{5.7.3}$$

Then equation (2.7.2) can be rewritten as follows:

$$Lx = \bar{N}x. \tag{5.7.4}$$

Some inequalities: From inequality (5.1.1), as shown in section 2.2, we have

$$||Nx - Ny|| \leq k_o\ |||x-y||| \tag{5.7.5}$$

for all $x,y \in D(N)$ where $|||\cdot|||$ is the norm on $H^{n-1}(J)$.

We define

$$\theta_m = \sqrt{b-a}\ \Big[\sum_{i=0}^{n-2} \Big(\sup_{t \in J} \int_a^b \Big(\frac{\partial^i K_m(t,s)}{\partial t^i}\Big)^2 ds\Big)^{1/2}\Big]$$

$$+\Big(\int_a^b \int_a^b \Big(\frac{\partial^{n-1} K_m(t,s)}{\partial t^{n-1}}\Big)^2 ds\ dt\Big)^{1/2} \tag{5.7.6}$$

and

$$\bar{\theta}_m = \max_{i=0,\ldots,n-1} \Big(\sup_{t \in J} \int_a^b \Big(\frac{\partial^i K_m(t,s)}{\partial t^i}\Big)^2 ds\Big)^{1/2} \tag{5.7.7}$$

where $K_m(\cdot,\cdot)$ is the function defined by (5.4.8).

As we saw in section 2.2, we have both θ_m and $\bar{\theta}_m \to 0$ as $m \to \infty$, and

$$|||\int_a^b K_m(\cdot,s)x(s)ds||| \leq \theta_m||x|| \tag{5.7.8}$$

and

$$\mu\Big(\int_a^b K_m(\cdot,s)x(s)ds\Big) \leq \bar{\theta}_m\ ||x|| \tag{5.7.9}$$

for all $x \in S$.

Definition of sets V and $\tilde{S}_o$:

Let us consider the Banach space $H^{n-1}(J)$. We note that $\tilde{H}^{n-1}(J)$ is a linear manifold of $H^{n-1}(J)$. We remember that μ on $\tilde{H}^{n-1}(J)$ is given by

$$\mu(x) = \max\left(\max_{i=0,\ldots,n-2} \sup_{t\in J} |x^{(i)}(t)|, \operatorname{ess.sup}_{t\in J} |x^{(n-1)}(t)|\right)$$

for all $x \in \tilde{H}^{n-1}(J)$.

We consider the m+n-dimensional space S_o given by (5.3.4). Clearly, $S_o \subset D(L) \subset H^n(J)$. We choose $x_o \in S_o$ such that $\beta = \mu(x_o) < R$ where $R = \min_{i=0,\ldots,n-1} R_i$. Here R_is are the constants in assumption (ii) of section 5.1. Let $z_o = F(I-P_m)Nx_o$, and let e and $\bar{e}$ be real constants such that

$$|||z_o||| \leq e \quad \text{and} \quad \mu(z_o) \leq \bar{e}. \tag{5.7.10}$$

Let c, d, r and $\bar{R}$ be real numbers such that

$$0<c<d,\ 0<r<R,\ c+e<d,\ \bar{R}+\beta\leq R, \text{ and } r+\bar{e} < \bar{R}. \tag{5.7.11}$$

The sets V and $\tilde{S}_o$ in $\tilde{H}^{n-1}(J)$ are defined as follows:

$$V = \{x \in S_o: |||x-x_o||| \leq c,\ \mu(x-x_o) \leq r\}, \tag{5.7.12}$$

and

$$\tilde{S}_o = \{x \in \tilde{H}^{n-1}(J): |||x-x_o||| \leq d,\ \mu(x-x_o) \leq \bar{R}\}. \tag{5.7.13}$$

Clearly, $x_o \in V \subset \tilde{S}_o \subset D(N)$. Moreover, V and $\tilde{S}_o$ are closed, bounded and convex subsets of $H^{n-1}(J)$. Also, V is a closed and bounded subset of S (see the section 2.3).

Operator T, and sets A(x*) and A and certain reduction:

For each $x^* \in V$, let T be the operator on $\tilde{S}_o$ defined by

$$Tx = x^* + H(\bar{I}-\bar{P}_m)\,\bar{N}x, \; x \in \tilde{S}_o. \tag{5.7.14}$$

By (5.7.3) and (5.6.2), (5.7.14) can be written as

$$Tx = x^* + F(I-P_m)Nx. \tag{5.7.15}$$

Thus T is well defined on $\tilde{S}_o$. We note that every fixed point of T belong to $D(\bar{N})$.

For each $x^* \in V$, the set $A(x^*)$ is defined by

$$A(x^*) = \{x \in \tilde{S}_o : x = Tx\}. \tag{5.7.16}$$

We denote by

$$A = \bigcup_{x^* \in V} A(x^*). \tag{5.7.17}$$

Suppose $A(x^*)$ is non-empty. Then there exists $\hat{x} \in \tilde{S}_o$ such that

$$\hat{x} = T\hat{x} = x^* + H\,(\bar{I}-\bar{P}_m)\bar{N}\hat{x}.$$

Clearly, $\hat{x} \in D(L)$ and by Theorem 5.5(iv) we have

$Q_m \hat{x} = x^*$.

Therefore, $\quad L\hat{x} = LQ_m\hat{x} + LH(\bar{I}-\bar{P}_m)\,\bar{N}\hat{x}.$

Using parts (ii) and (iii) of theorem 5.5, we get

$$L\hat{x} - \bar{N}\hat{x} = \bar{P}_m\,(L\hat{x} - \bar{N}\hat{x}). \tag{5.7.18}$$

Hence, $\hat{x} \in \tilde{S}_o$ is a solution of (5.1.5) - (5.1.6) if

$$\bar{P}_m(L\hat{x}-\bar{N}\hat{x}) = 0. \tag{5.7.19}$$

Equation (5.7.19) is called the bifurcation equation of order m. We notice that if A is non-empty, then $Lx-\bar{N}x = \bar{P}_m(Lx-\bar{N}x)$ on A.

The following theorem gives that $Lx-\bar{N}x = \bar{P}_m(Lx-\bar{N}x)$ on A.

THEOREM 5.7. Let the assumptions (i) - (iv) of section 5.1 and conditions (5.7.10) and (5.7.11) be satisfied. Let 'm' be sufficiently large such that

$$\theta_m k_o < 1, \; c+e \leq (1-\theta_m k_o)d \text{ and } r+\bar{e} \leq \bar{R}-\bar{\theta}_m k_o d. \quad (5.7.20)$$

Then for each $x^* \in V$ the set $A(x^*)$ is singleton. Moreover, $Lx - \bar{N}x = \bar{P}_m(Lx-\bar{N}x)$ on the set A.

If we take the expression (5.7.15) for T, then the proof of the theorem is similar to the proof of theorem 2.2 and can be proved easily.

By Theorem 5.7, the original MPBVP (5.1.5)-(5.1.6) is reduced to the equivalent bifurcation equation (5.7.19). In the rest of the chapter, we solve the bifurcation equation (5.7.19).

Solution of the bifurcation equation:

Throughout this section the conditions of Theorem 5.7 and assumption (v) of Section 5.1 are assumed to be valid. By Theorem 5.7, we know that for each $x^* \in V$ the set $A(x^*)$ is singleton. Let $A(x^*) = \hat{x}$. Thus for each $x^* \in V$ there exists a unique element $\hat{x} \in A \subset \tilde{S}_o$ such that

$$\hat{x} = T\hat{x} = x^* + H(\bar{I} - \bar{P}_m)\ \bar{N}\hat{x}$$
$$= x^* + F(I - P_m)\ N\hat{x}.$$

As we did in section 2.7, we can show that $\hat{x}$ vary with x^* continuously in $H^{n-1}(J)$. Under the hypothesis of theorem 5.7, let $\Gamma : V \to D(L) \cap \tilde{S}_o$ be the continuous operator defined by $\Gamma(x^*) = \hat{x}$ where $\hat{x}$ is the unique element in $\tilde{S}_o$ which is a fixed point of the operator T corresponding to x^*. We note that $\bar{P}_m(L\Gamma x^* - \bar{N}\Gamma x^*)$ is an operator mapping V into the subspace $< \omega_1, \omega_2, \ldots, \omega_m > \times R^k$ of the space Y.

We now state the following lemma.

<u>LEMMA</u> 5.1 . Suppose the assumptions of theorem 5.7 are satisfied. Let $\{x_m^*\}$ be any sequence contained in V. Suppose $\{x_m^*\}$ converges to x^* in the topology of S. Then the sequence $\{\Gamma x_m^*\}$ converges to Γx^* in the topology of $H^{n-1}(J)$.

Proof of the above lemma is similar to the proof of lemma 2.4 and can be verified easily.

We need the following lemma for our discussions.

<u>LEMMA</u> 5.2 . Let the assumptions of theorem 5.7 be valid. Suppose x^* and $y^* \in V$. Then

$$\mu(\Gamma x^* - \Gamma y^*) \leq \mu(x^* - y^*) + \bar{\theta}_m\ k_o\ |||\Gamma x^* - \Gamma y^*|||. \qquad (5.7.21)$$

<u>Proof</u> : By theorem 5.7, we have

$$\Gamma x^* \neq \cancel{Tx^*} = x^* + H(\bar{I} - \bar{P}_m)\ \bar{N}\Gamma x^*$$

and

$$\Gamma y^* \neq \cancel{Ty^*} = y^* + H(\bar{I} - \bar{P}_m)\ \bar{N}\Gamma y^*.$$

We also have

$$\Gamma x^* = x^* + F(I - P_m)\, N\Gamma x^*$$

and

$$\Gamma y^* = y^* + F(I - P_m)\, N\Gamma y^* \quad \text{(see (5.7.15))}.$$

Therefore,

$$\begin{aligned}
\mu(\Gamma x^*-\Gamma y^*) &\leq \mu(x^*-y^*) + \mu(F(I-P_m)(N\Gamma x^*-N\Gamma y^*)) \\
&= \mu(x^*-y^*) + \mu\left(\int_a^b K_m(\cdot,s)\left[(N\Gamma x^*-N\Gamma y^*)(s)\right] ds\right) \\
&\qquad\qquad \text{(by (5.4.9))} \\
&\leq \mu(x^*-y^*) + \bar{\theta}_m\, \|N\Gamma x^*-N\Gamma y^*\| \quad \text{(by (5.7.9))} \\
&\leq \mu(x^*-y^*) + \bar{\theta}_m\, k_o|||\Gamma x^*-\Gamma y^*||| \quad \text{(by (5.7.5))}.
\end{aligned}$$

This completes the proof of the lemma.

The next theorem is an immediate consequence of theorem 5.7.

THEOREM 5.8 . Let assumptions of theorem 5.7 be valid. If there exists an element $x^* \in V$ such that

$$\bar{P}_m(L\Gamma x^* - \bar{N}\Gamma x^*) = 0, \tag{5.7.22}$$

then the element $\hat{x} = \Gamma x^*$ is a solution of the MPBVP (5.1.5) - (5.1.6). Furthermore, $Q_m x = x^*$, $|||\hat{x}-x^*||| \leq d$, $\mu(\hat{x}-x_o) \leq \bar{R}$.

In theorem 5.8, the problem of solving (5.1.5) - (5.1.6) has been reduced to the problem of solving the equation (5.7.22). Below, we solve the equation (5.7.22) under certain additional assumptions.

Let $\psi : D(L) \cap \tilde{S}_o \to \langle \omega_1, \ldots \omega_m \rangle \times R^k$ be the operator defined by

$$\psi x = \bar{P}_m(Lx - \bar{N}x) \tag{5.7.23}$$

for all $x \in D(L) \cap \tilde{S}_o$.

We note that $V \subset D(L) \cap \tilde{S}_o$. Let x^* and $y^* \in V$. Then

$$\psi \Gamma x^* - \psi \Gamma y^* = \bar{P}_m(L\Gamma x^* - L\Gamma y^* - \bar{N}\Gamma x^* + \bar{N}\Gamma y^*)$$

$$= \bar{P}_m(T_1(\tau)(\Gamma x^* - \Gamma y^*) - N\Gamma x^* + N\Gamma y^*, -f(\Gamma x^*) + f(\Gamma y^*))$$

(by (5.5.1) and (5.7.3))

$$= (P_m[T_1(\tau)(\Gamma x^* - \Gamma y^*) - N\Gamma x^* + N\Gamma y^*],$$
$$- f(\Gamma x^*) + f(\Gamma y^*))$$

(by the definition of $\bar{P}_m$).

Therefore,

$$||\psi \Gamma x^* - \psi \Gamma y^*||_Y = ||P_m[T_1(\tau)(\Gamma x^* - \Gamma y^*) - N\Gamma x^* + N\Gamma y^*]||$$
$$+ |f(\Gamma x^*) - f(\Gamma y^*)|$$

$$\leq || \sum_{i=1}^{m} (T_1(\tau)(\Gamma x^* - \Gamma y^*), \omega_i) \omega_i||$$

$$+ ||N\Gamma x^* - N\Gamma y^*|| + |f(\Gamma x^*) - f(\Gamma y^*)|$$

(by triangle and Bessel's inequalities)

$$= || \sum_{i=1}^{m} (\Gamma x^* - \Gamma y^*, T_o(\tau^*) \omega_i) \omega_i|| + ||N\Gamma x^* - N\Gamma y^*||$$
$$+ |f(\Gamma x^*) - f(\Gamma y^*)|$$

$$\leq \sum_{i=1}^{m} ||\Gamma x^* - \Gamma y^*|| \; ||T_o(\tau^*)\; \omega_i|| + k_o|||\Gamma x^* - \Gamma y^*|||$$

$$+ |f(\Gamma x^*) - f(\Gamma y^*)|$$

(by Schwartz inequality and relation (5.7.5))

$$\leq (\sum_{i=1}^{m} ||T_o(\tau^*)\; \omega_i|| + k_o)|||\Gamma x^* - \Gamma y^*||| + |f(\Gamma x^*) - f(\Gamma y^*)|$$

(since $||\cdot|| \leq |||\cdot|||$)

$$\leq (\sum_{i=1}^{m} ||T_o(\tau^*)\; \omega_i|| + k_o)|||\Gamma x^* - \Gamma y^*|||$$

$$+ \nu(\sum_{j=1}^{k} |f_j(\Gamma x^*) - f_j(\Gamma y^*)|^2)^{1/2}$$

(see (5.7.1)).

But by the assumption (v) of section 5.1, we have

$$\nu(\sum_{j=1}^{k} |f_j(\Gamma x^*) - f_j(\Gamma y^*)|^2)^{1/2}$$

$$\leq \nu(\sum_{j=1}^{k} \ell_j^2)^{1/2} \max_{i=0,\ldots,n-1} \sup_{t \in J} |(\Gamma x^*)^{(i)} - (\Gamma y^*)^{(i)}|)$$

$$= \nu(\sum_{j=1}^{k} \ell_j^2)^{1/2} \mu(\Gamma x^* - \Gamma y^*). \qquad (5.7.24)$$

Thus under the assumptions of theorem 5.7 and assumption (v) of section 5.1, we have

$$||\psi\Gamma x^* - \psi\Gamma y^*||_Y \leq (\sum_{i=1}^{m} ||T_o(\tau^*)\; \omega_i|| + k_o)\; |||\Gamma x^* - \Gamma y^*|||$$

$$+ \nu(\sum_{j=1}^{k} \ell_j^2)^{1/2} \; \mu(\Gamma x^* - \Gamma y^*).$$

We also know that if x^* converges to y^* in the topology of S, then x^* converges to y^* in the topology of $H^n(J)$. Hence, by lemma 5.1 and lemma 5.2, it readily follows that $||\psi\Gamma x^* - \psi\Gamma y^*||_Y \to 0$ as x^* converges to y^* in the topology of S. Thus

$$\psi\Gamma : V \subset S \to \langle \omega_1, \ldots, \omega_m \rangle \times R^k \subset Y$$

is continuous.

Similarly, we can show that

$$\psi : V \subset S \to \langle \omega_1, \ldots, \omega_m \rangle \times R^k \subset Y$$

is continuous.

Also, by (5.7.23), the equation (5.7.22) can be rewritten as

$$\psi\Gamma x^* = 0. \tag{5.7.25}$$

Since Γ is defined implicitly, the existence of a solution of $\psi\Gamma x^* = 0$ is better studied through the operator ψ restricted to V.

The following lemma relates these two operators.

<u>LEMMA</u> 5.3 . Let the assumptions of section 5.1 and conditions (5.7.10) and (5.7.11) be satisfied. Suppose the relations (5.7.20) are valid. Then for each $x^* \in V$ we have

$$||\psi\Gamma x^* - \psi x^*||_Y \leq (\theta_m k_o d + e) k_o + \nu \left(\sum_{j=1}^{k} \ell_j^2 \right)^{1/2} (\bar{\theta}_m k_o d + \bar{e}). \tag{5.7.26}$$

<u>Proof</u> : Suppose $x^* \in V$. Let $x = \Gamma x^*$. Then $x \in D(L) \cap \tilde{S}_o$, $Q_m\, x = x^*$ and $\bar{P}_m\, Lx = \bar{P}_m\, Lx^*$. So

$$\psi\Gamma x^* - \psi x^* = \bar{P}_m(\bar{N}x^* - \bar{N}\Gamma x^*)$$

$$= (P_m(Nx^* - N\Gamma x^*),\ f(x^*) - f(\Gamma x^*)).$$

Hence,

$$||\psi\Gamma x^* - \psi\, x^*||_Y \leq ||Nx^* - N\Gamma x^*|| + |f(x^*) - f(\Gamma x^*)|$$

$$\leq k_o\ |||x^* - \Gamma x^*||| + |f(x^*) - f(\Gamma x^*)|$$

(by (5.7.5))

$$\leq k_o\ |||x^* - \Gamma x^*|| + \nu(\sum_{j=1}^{k} \ell_j^2)^2\ \mu(x^* - \Gamma x^*)$$

(similar to (5.7.24)). (5.7.27)

But,

$$\Gamma x^* - x^* = F(I - P_m)\, N\Gamma x^* \quad \text{(see (5.7.15))}$$

$$= F(I - P_m)(N\Gamma x^* - Nx_o) + z_o$$

$$= \int_a^b K_m(\cdot,s)((N\Gamma x^*)(s) - (Nx_o)(s))\, ds + z_o.$$

Therefore,

$$|||\Gamma x^* - x^*||| \leq |||\int_a^b K_m(\cdot,s)((N\Gamma x^*)(s) - (Nx_o)(s))\, ds\ ||| + |||z_o|||$$

$$\leq \theta_m\ ||N\Gamma x^* - Nx_o|| + e \quad \text{(by (5.7.8) and (5.7.10))}$$

$$\leq \theta_m\, k_o\ |||\Gamma x^* - x_o||| + e \quad \text{(by (5.7.5))}$$

$$\leq \theta_m\, k_o\, d + e \qquad \text{(by (5.7.13)).} \quad (5.7.28)$$

Also,

$$\mu(\Gamma x^* - x^*) \le \mu\left(\int_a^b K_m(\cdot,s)((N\Gamma x^*)(s)-(Nx_o)(s))ds\right) + \mu(z_o)$$

$$\le \bar{\theta}_m \, ||N\Gamma x^* - Nx_o|| + \bar{e} \quad \text{(by (5.7.9) and (5.7.10))}$$

$$\le \bar{\theta}_m \, k_o \, |||\Gamma x^* - x_o||| + \bar{e} \quad \text{(by (5.7.5))}$$

$$\le \bar{\theta}_m \, k_o \, d + \bar{e} \qquad \text{(by (5.7.13)).} \qquad (5.7.29)$$

Therefore, from (5.7.27), (5.7.28) and (5.7.29), we get

$$||\psi \Gamma x^* - \psi x^*||_Y \le k_o(\theta_m \, k_o d + e) + \nu\left(\sum_{j=1}^{k} \ell_j^2\right)^{1/2} (\bar{\theta}_m \, k_o \, d + \bar{e}).$$

This completes the proof of the lemma.

We use the above lemma to determine the conditions on $\psi|V$ which guarantee that the equation (5.7.25) is solvable. Since ψ and $\psi\Gamma$ both restricted to V map a finite-dimensional space into another finite-dimensional space, we shall define a map which takes one coefficient space into the other.

For the above said purpose, we apply the Gram-Schmidt process to the elements $F\omega_1,\ldots,F\omega_m$ to obtain orthonormal elements $\eta_1,\ldots,\eta_m$. Let $\bar{m} = n + m$, and let $\bar{\bar{m}} = m+k$. We remember that $k \le n$ is the number boundary conditions. Let $E^{\bar{m}}$ be a copy of Euclidean $\bar{m}$-space where we represent each point $\xi \in E^{\bar{m}}$ as an $\bar{m}$-tuple : $\xi = (b_1,\ldots,b_n, c_1,\ldots,c_m)$. Also, let $E^{\bar{\bar{m}}}$ be a copy of Euclidean $\bar{\bar{m}}$-space where we represent each point $v \in E^{\bar{\bar{m}}}$ as an $\bar{\bar{m}}$-tuple: $v = (u_1,\ldots,u_m, \alpha_1,\ldots,\alpha_k)$.

We define two operators

$$\Gamma_1 : E^{\bar{m}} \to S_o \quad \text{and} \quad \Gamma_2 : \langle \omega_1, \ldots, \omega_m \rangle \times R^k \to E^{\bar{\bar{m}}} \text{ by}$$

$$\Gamma_1 (b_1, \ldots, b_n, c_1, \ldots, c_m) = \sum_{i=1}^{n} b_i \phi_i + \sum_{i=1}^{m} c_i \eta_i \tag{5.7.30}$$

and

$$\Gamma_2 \left(\sum_{i=1}^{m} u_i \omega_i, \alpha_1, \ldots, \alpha_k \right) = (u_1, \ldots, u_m, \alpha_1, \ldots, \alpha_k). \tag{5.7.31}$$

Clearly, Γ_1 and Γ_2 are isomorphisms. Let $\xi_o \in E^{\bar{m}}$ be the element with $\Gamma_1(\xi_o) = x_o$, and let $\Psi : E^{\bar{m}} \to E^{\bar{\bar{m}}}$ be the operator defined by

$$\Psi = \Gamma_2 \psi \Gamma_1 \tag{5.7.32}$$

Let us choose a number $\varepsilon > 0$ such that the set

$$U = \{ \xi \in E^{\bar{m}} : | \xi - \xi_o | \leq \varepsilon \} \tag{5.7.33}$$

is mapped by Γ_1 into the set V. The existence of such ε is not difficult to show (see appendix 3). Under the hypothesis of theorem 5.7 and assumption (v) of section 5.1, we observe that $\Gamma_2 \psi \Gamma_1$ and $\Gamma_2 \psi \Gamma \Gamma_1$ map the ball $U \subset E^{\bar{m}}$ continuously into $E^{\bar{\bar{m}}}$. This is used in the following theorems to establish the existence of a solution to the equation (5.7.25).

THEOREM 5.9 . Suppose $\Psi(\xi_o) = 0$ and the following conditions are satisfied :

(i) The map Ψ has first order continuous partial derivatives in the interior of U.

(ii) The Jacobian matrix for Ψ has rank $\bar{\bar{m}}$ at ξ_o.

Then there exists a number $\delta > 0$ and a continuous map Λ such that the set

$$\bar{\Omega} = \{ v \in E^{\bar{\bar{m}}} : |v| \leq \delta \} \tag{5.7.34}$$

is a subset of $\Psi(U)$ and $\Lambda : \bar{\Omega} \to U$ with $\Psi \Lambda (u) = u$ for all $u \in \bar{\Omega}$.

For a proof of the lemma see the reference given for the proof of theorem 0.2.

<u>THEOREM</u> 5.10 . Let the assumptions of section 5.1 and conditions (5.7.10) and (5.7.11) be satisfied. Suppose the relations (5.7.20) are valid. Let the assumptions of theorem 5.9 be valid and let

$$k_o(\theta_m\, k_o\, d + e) + \nu\Big(\sum_{j=1}^{k} \ell_j^2\Big)^{1/2} (\bar{\theta}_m\, k_o\, d + \bar{e}) < \delta$$

where δ is the number in (5.7.34). Then there exists an $x^* \in V$ such that $\psi \Gamma x^* = 0$. Moreover, $\hat{x} = \Gamma x^*$ is a solution of the original MPBVP (5.1.5) - (5.1.6). Also, $Q_m\, \hat{x} = x^*, |||x-x_o||| \leq d$ and $\mu(\hat{x}-x_o) \leq \bar{R}$.

<u>Proof</u> : Let us consider the map $\Gamma_2\ \psi \Gamma \Gamma_1\ \Lambda : \bar{\Omega} \to E^m$. Take $v \in \bar{\Omega}$ and let $x^* = \Gamma_1\ \Lambda(v)$. Then $x^* \in V$ and $\Gamma_2\ \psi\ x^* = \Gamma_2\ \psi\ \Gamma_1\ \Lambda(v) = v$. Moreover,

$$|\Gamma_2\ \psi\ \Gamma\Gamma_1\Lambda(v) - v| = |\Gamma_2\ \psi\ \Gamma\Gamma_1\ \Lambda(v) - \Gamma_2\ \psi\ x^*|$$

$$= |\Gamma_2\ \psi\ \Gamma x^* - \Gamma_2\ \psi\ x^*|.$$

But,

$$\psi\Gamma x^* - \psi x^* = (\sum_{i=1}^{m} u_i \ \omega_i, \alpha_1,\ldots,\alpha_k) \text{ for some}$$

$$(u_1,\ldots,u_m, \alpha_1,\ldots,\alpha_k) \in E^{\bar{\bar{m}}}.$$

Therefore,

$$|\Gamma_2 \psi \ \Gamma x^* - \Gamma_2 \psi \ x^*| = (u_1^2 + \cdots + u_m^2 + \alpha_1^2 + \cdots + \alpha_k^2)^{1/2}$$

$$\leq (u_1^2+\cdots+u_m^2)^{1/2}+(\alpha_1^2+\cdots+\alpha_k^2)^{1/2}$$

(since $(a^2+b^2)^{1/2} \leq a+b$ for positive a and b)

$$= ||\psi\Gamma x^* - \psi x^*||_Y .$$

Hence,

$$|\Gamma_2 \psi \ \Gamma\Gamma_1 \Lambda(v) - v| \leq ||\psi\Gamma x^* - \psi x^*||_Y$$

$$\leq k_o(\theta_m k_o d+e)+\nu(\sum_{j=1}^{k} \ell_j^2)^{1/2}(\bar{\theta}_m k_o d+\bar{e})$$

(by lemma 5.3)

$$< \delta .$$

Thus

$$|\Gamma_2 \psi \ \Gamma\Gamma_1 \Lambda(v) - v| < \delta \quad \text{for } v \in \bar{\Omega}.$$

Hence by theorem 0.3 (iii) we have $d(\Gamma_2 \ \psi \ \Gamma\Gamma_1 \Lambda(v), \mathring{\Omega}, 0) = 1$. Therefore by theorem 0.3 (iv) there exists an element $v \in \mathring{\Omega}$ such that $\Gamma_2 \ \psi \ \Gamma\Gamma_1 \ \Lambda(v) = 0$. Setting $x^* = \Gamma_1 \Lambda(v)$, we have $x^* \in V$ and $\psi\Gamma x^* = 0$. Obviously, by theorem 5.8, $\hat{x} = \Gamma x^*$ is a solution of the original MPBVP (5.1.5) - (5.1.6). Moreover,

$Q_m \hat{x} = x^*$. Since $\hat{x} \in \tilde{S}_0$, we clearly have $|||\hat{x}-x_0||| \leq d$ and $\mu(\hat{x}-x_0) \leq \bar{R}$.

REMARK 5.1 . As mentioned in remark 2.1, if X is of the form $X(t,x_0,\ldots,x_q)$ where $q \leq n-1$, then from our analysis of this chapter we observe that it is enough to consider the space $H^q(J)$. In this case, $|||\cdot|||$ will be the corresponding norm on $H^q(J)$. Also, the quantities θ_m and $\bar{\theta}_m$ are to be defined subsequently. Actually, θ_m and $\bar{\theta}_m$ takes the following form :

$$\theta_m = \sqrt{b-a}\ \left(\sum_{i=0}^{q-1} \left(\sup_{t \in J} \int_a^b \left(\frac{\partial^i K_m(t,s)}{\partial t^i}\right)^2 ds\right)^{1/2}\right)$$

$$+ \left(\int_a^b \int_a^b \left(\frac{\partial^q K_m(t,s)}{\partial t^q}\right)^2 ds\ dt\right)^{1/2}$$

$$\bar{\theta}_m = \max_{i=0,\ldots,q} \left(\sup_{t \in J} \int_a^b \left(\frac{\partial^i K_m(t,s)}{\partial t^i}\right)^2 ds\right)^{1/2}.$$

Also, in the case μ will be the corresponding function on $\tilde{H}^{q-\cancel{1}}(J)$.

REMARK 5.2 . From our analysis, it is also clear that there is no need of an infinite sequence of projections. Indeed, what we need is that 'm' should be sufficiently large such that the conditions of theorem 5.7 are satisfied.

5.8. AN ILLUSTRATIVE EXAMPLE

We make use of the theory developed in the earlier sections of this chapter and prove the existence of a solution to the following second-order nonlinear differential equation

with nonlinear three-point boundary conditions :

$$x^{(2)} + x = \frac{x^3}{2},$$

$$\frac{1}{8}\,(x^{(1)}(0) - x^{(1)}(\pi))^3 - x^{(1)}(2\pi) = 0 \qquad (5.8.1)$$

$$\frac{1}{8}\,(x(0) + x(2\pi))^3 + x(\pi) = 0$$

over the interval $J = [0,2\pi]$.

Let $\tau x = x^{(2)} + x$.

As usual, the operator $T_1(\tau)$ is defined as follows :

$$D(T_1(\tau)) = H^2(J),\ T_1(\tau)x = \tau x = x^{(2)} + x.$$

Clearly, the adjoint of $T_1(\tau) = T_o(\tau^*)$. We note that $D(T_o(\tau^*)) = H_o^2(J)$ and $T_o(\tau^*)y = y^{(2)} + y$. We can check that the functions ϕ_1 and ϕ_2 given by $\phi_1(t) = \frac{\cos t}{\sqrt{\pi}}$ and $\phi_2(t) = \frac{\sin t}{\sqrt{\pi}}$ form an orthonormal basis for $N(T_1(\tau))$. Then

$$\det \Phi(t) = \begin{vmatrix} \frac{\cos t}{\sqrt{\pi}} & \frac{\sin t}{\sqrt{\pi}} \\ -\frac{\sin t}{\sqrt{\pi}} & \frac{\cos t}{\sqrt{\pi}} \end{vmatrix} = \frac{1}{\pi}\,(\cos^2 t + \sin^2 t) = \frac{1}{\pi},$$

$$W_1(t) = \begin{vmatrix} 0 & \frac{\sin t}{\sqrt{\pi}} \\ 1 & \frac{\cos t}{\sqrt{\pi}} \end{vmatrix} = -\frac{\sin t}{\sqrt{\pi}},$$

$$W_2(t) = \begin{vmatrix} \frac{\cos t}{\sqrt{\pi}} & 0 \\ -\frac{\sin t}{\sqrt{\pi}} & 1 \end{vmatrix} = \frac{\cos t}{\sqrt{\pi}}.$$

Therefore,

$$G(t,s) = \sum_{i=1}^{2} \frac{\phi_i(t)\, W_i(s)}{\det\, \Phi(s)}$$

$$= \left(-\frac{\cos\, t}{\pi}\,\sin\, s + \frac{\sin\, t\, \cos\, s}{\pi}\right)\pi$$

$$= \sin\, t\, \cos\, s - \cos\, t\, \sin\, s. \tag{5.8.2}$$

Determination of ψ_1 and ψ_2 :

We have

$$\psi_1(t) = -\int_t^{2\pi} \phi_1(s)\, G(s,t)\, ds$$

$$= -\int_t^{2\pi} \frac{\cos\, s}{\sqrt{\pi}}\, (\cos\, t\, \sin\, s - \cos\, s\, \sin\, t)\, ds.$$

Elementary integration yields that

$$\psi_1(t) = -\frac{1}{4\sqrt{\pi}}\left[\cos\, t\,(\cos\, 2t-1) - \sin\, t\,(4\pi-2t-\sin\, 2t)\right].$$

We also have

$$\psi_2(t) = -\int_t^{2\pi} \phi_2(s)\, G(s,t)\, ds$$

$$= -\int_t^{2\pi} \frac{\sin\, s}{\sqrt{\pi}}\left[\cos\, t\, \sin\, s - \cos\, s\, \sin\, t\right] ds.$$

Again, elementary integration yields that

$$\psi_2(t) = -\frac{1}{4\sqrt{\pi}}\left[\cos\, t(4\pi-2t+\sin 2t)+\sin t(1-\cos\, 2t)\right]$$

Determination of $K(t,s)$:

We have

$$\sum_{i=1}^{2} \phi_i(t)\, \psi_i(s)$$

$$= -\frac{\cos t}{4\pi}\left[\cos s\,(\cos 2s - 1) - \sin s\,(4\pi - 2s - \sin 2s)\right]$$

$$- \frac{\sin t}{4\pi}\left[\cos s(4\pi - 2s + \sin 2s) + \sin s(1 - \cos 2s)\right].$$

Simple calculation yields that

$$\sum_{i=1}^{2} \phi_i(t)\, \psi_i(s) = -\frac{1}{4\pi}\Big[(2s \sin s - 4\pi \sin s) \cos t$$

$$+ (2 \sin s - 2s \cos s + 4\pi \cos s)\sin t\Big].$$

Therefore,

$$K(t,s) = \begin{cases} -\dfrac{1}{2\pi}\Big[(s \sin s - 2\pi \sin s) \cos t \\ \quad + (\sin s - s \cos s + 2\pi \cos s) \sin t\Big] \\ \quad + \sin t \cos s - \cos t \sin s, \; 0 \leq s \leq t \leq 2\pi, \\ -\dfrac{1}{2\pi}\Big[(s \sin s - 2\pi \sin s) \cos t \\ \quad + (\sin s - s \cos s + 2\pi \cos s) \sin t\Big], \\ \qquad 0 \leq t \leq s \leq 2\pi. \end{cases} \tag{5.8.3}$$

Let the function ω_1 be defined by

$$\omega_1(t) = \frac{1}{\sqrt{2\pi}} \;(\cos 2t - \cos 4t). \tag{5.8.4}$$

We notice that $\omega_1 \in D(T_o(\tau^*))$. Clearly, ω_1 is a normalized vector.

The projection P_1 is defined by

$$P_1 x = (x, \omega_1)\, \omega_1. \tag{5.8.5}$$

Determination of $K_1(t,s)$:

We have

$$K_1(t,s) = K(t,s) - \frac{1}{2\pi}\left(\left(\int_0^{2\pi} K(t,\xi)(\cos 2\xi - \cos 4\xi)\,d\xi\right)(\cos 2s - \cos 4s)\right).$$

But, elementary integration yields that

$$\int_0^{2\pi} K(t,s)(\cos 2s - \cos 4s)\,ds = \left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right)$$

(we use the expression (5.8.3) for $K(t,s)$).

Hence,

$$K_1(t,s) = \begin{cases} -\frac{1}{2\pi}\Big[(s \sin s - 2\pi \sin s)\cos t + (\sin s - s\cos s + 2\pi \cos s)\sin t \\ \quad + \left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right)(\cos 2s - \cos 4s)\Big] \\ \quad + \sin t \cos s - \cos t \sin s, \quad 0 \le s \le t \le 2\pi, \\ -\frac{1}{2\pi}\Big[(s \sin s - 2\pi \sin s)\cos t + (\sin s - s\cos s + 2\pi \cos s)\sin t \\ \quad + \left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right)(\cos 2s - \cos 4s)\Big], \\ \quad 0 \le t \le s \le 2\pi. \end{cases} \tag{5.8.6}$$

Determination of θ_1 and $\bar{\theta}_1$:

After elementary and lengthy integrations, we get

$$\int_0^{2\pi} K_1(t,s)^2\, ds = \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} - \frac{1}{8\pi} + \frac{t \sin 2t}{4\pi} - \frac{\sin 2t}{4}$$

$$- \frac{1}{2\pi}\left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right)^2, \quad t \in J. \quad (5.8.7)$$

Since

$\frac{t \sin 2t}{4\pi} \leq \frac{t}{4\pi}$, $-\frac{\sin 2t}{4} \leq \frac{1}{4}$ over the interval $[0,2\pi]$ and

$\left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right)^2 \geq 0$,

we have

$$\int_0^{2\pi} K_1(t,s)^2\, ds \leq \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} - \frac{1}{8\pi} + \frac{t}{4\pi} + \frac{1}{4} . \quad (5.8.8)$$

Let $\alpha(t) = \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} - \frac{1}{8\pi} + \frac{t}{4\pi} + \frac{1}{4}$. (5.8.9)

Then $\alpha(0) = \frac{\pi}{3} + \frac{1}{4} - \frac{1}{8\pi}$, and $\alpha(2\pi) = \frac{\pi}{3} - \frac{1}{8\pi} + \frac{3}{4}$.

Moreover, $\alpha^{(1)}(t) = \frac{t}{2\pi} - \frac{1}{2} + \frac{1}{4\pi}$.

The point $t = \frac{2\pi-1}{2}$ is a solution of $\alpha^{(1)}(t) = 0$.

But, $\alpha(\frac{2\pi-1}{2}) = \frac{\pi}{3} + \frac{1}{2} - \frac{\pi}{4} - \frac{3}{16\pi}$.

Therefore, $\sup_{t \in [0,2\pi]} \alpha(t) = \frac{\pi}{3} + \frac{3}{4} - \frac{1}{8\pi} \leq \frac{6497}{3696}$. (5.8.10)

Hence, from (5.8.8), (5.8.9) and (5.8.10), we get

$$\left(\sup_{s \in [0,2\pi]} \int_0^{2\pi} K_1(t,s)^2 \, ds\right)^{1/2} \leq \left(\frac{6497}{3696}\right)^{1/2} < 1.33. \qquad (5.8.11)$$

Differentiating the expression (5.8.6) with respect to t, we get

$$\frac{\partial K_1(t,s)}{\partial t} = \begin{cases} -\frac{1}{2\pi}\Big[-(s \sin s - 2\pi \sin s)\sin t + (\sin s - s \cos s \\ \qquad + 2\pi \cos s)\cos t \\ \qquad + (\frac{2}{3} \sin 2t - \frac{4}{15} \sin 4t)(\cos 2s - \cos 4s)\Big] \\ \qquad + \cos t \cos s + \sin t \sin s, 0 \leq s \leq t \leq 2\pi, \\ -\frac{1}{2\pi}\Big[-(s \sin s - 2\pi \sin s)\sin t + (\sin s - s\cos s \\ \qquad + 2\pi \cos s) \cos t \\ \qquad + (\frac{2}{3} \sin 2t - \frac{4}{15} \sin 4t)(\cos 2s - \cos 4s)\Big], \\ \qquad 0 \leq t \leq s \leq 2\pi. \end{cases} \qquad (5.8.12)$$

After elementary and lengthy integrations, we get

$$\int_0^{2\pi} \left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds = \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} + \frac{3}{8\pi} + \frac{\cos 2t}{4\pi} + \frac{\sin 2t}{4} - \frac{t \sin 2t}{4\pi}$$
$$- \frac{1}{2\pi}\left(\frac{2 \sin 2t}{3} - \frac{4 \sin 4t}{15}\right)^2, \; t \in J. \qquad (5.8.13)$$

Since $\frac{\cos 2t}{4\pi} \leq \frac{1}{4\pi}$, $\frac{\sin 2t}{4} \leq \frac{1}{4}$, $\frac{-t \sin 2t}{4\pi} \leq \frac{t}{4\pi}$ over the interval $[0, 2\pi]$ and $(\frac{2 \sin 2t}{3} - \frac{4 \sin 4t}{15})^2 \geq 0$ for $t \in [0,2\pi]$, we get

$$\int_0^{2\pi} (\frac{\partial K_1(t,s)}{\partial t})^2 \, ds \leq \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} + \frac{3}{8\pi} + \frac{1}{4\pi} + \frac{1}{4} + \frac{t}{4\pi} . \quad (5.8.14)$$

Let

$$\beta(t) = \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{3} + \frac{3}{8\pi} + \frac{1}{4\pi} + \frac{1}{4} + \frac{t}{4\pi} . \quad (5.8.15)$$

We have

$$\beta(0) = \frac{\pi}{3} + \frac{5}{8\pi} + \frac{1}{4} \quad \text{and} \quad \beta(2\pi) = \frac{\pi}{3} + \frac{5}{8\pi} + \frac{3}{4} .$$

Moreover,

$$\beta^{(1)}(t) = \frac{t}{2\pi} - \frac{1}{2} + \frac{1}{4\pi} .$$

The point $t = \frac{2\pi-1}{2}$ is a root of $\beta^{(1)}(t)$.

Calculating, we get

$$\beta(\frac{2\pi-1}{2}) = \frac{\pi}{3} + \frac{1}{2} - \frac{\pi}{4} + \frac{9}{16\pi} .$$

Hence,

$$\sup_{t \in [0,2\pi]} \beta(t) = \frac{\pi}{3} + \frac{5}{8\pi} + \frac{3}{4} = \frac{7379}{3696} . \quad (5.8.16)$$

Therefore, from (5.8.14), (5.8.15) and (5.8.16), we get

$$(\sup_{t \in [0,2\pi]} \int_0^{2\pi} (\frac{\partial K_1(t,s)}{\partial t})^2 \, ds)^{1/2} = (\frac{7379}{3696})^{1/2} < 1.43. \quad (5.8.17)$$

Also, simple integration of (5.8.13) yields that

$$\left(\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\ dt\right)^{1/2} = \left(\frac{\pi^2}{3} + \frac{167}{225}\right)^{1/2} < 2.25. \quad (5.8.18)$$

From (5.8.11) and (5.8.18), we get

$$\theta_1 = \sqrt{2\pi}\ \left(\sup_{t\,\in\,[0,2\pi]}\int_0^{2\pi} K_1(t,s)^2\ ds\right)^{1/2} + \left(\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\ dt\right)^{\frac{1}{2}}$$

$$< (\sqrt{2\pi} \times 1.33) + 2.25 < 3.34 + 2.25 = 5.59.$$

Therefore,

$$\theta_1 < 5.59. \quad (5.8.19)$$

Also, from (3.8.11) and (3.8.17), we get

$$\bar{\theta}_1 = \max\left(\left(\sup_{t\,\in\,[0,2\pi]}\int_0^{2\pi} K_1(t,s)^2\ ds\right)^{1/2}, \left(\sup_{t\,\in\,[0,2\pi]}\int_0^{2\pi}\left(\frac{\partial K_1(t,s)}{\partial t}\right)^2 ds\right)^{1/2}\right)$$

$$< \max\ (1.33,\ 1.43) = 1.43.$$

Therefore,

$$\bar{\theta}_1 < 1.43. \quad (5.8.20)$$

Here $X(t,x,x^{(1)}) = \frac{x^3}{2}$. Therefore, $(Nx)(t) = \frac{(x(t))^3}{2}$.

Take $m = 1$.

We show that $H\omega_1$ is given by

$$(H\omega_1)(t) = \frac{1}{\sqrt{2\pi}}\left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right).$$

Normalizing the vector $H\omega_1$, we get

$$\eta_1 = \frac{15}{\sqrt{26\pi}} \left(\frac{\cos 4t}{15} - \frac{\cos 2t}{3}\right).$$

We have

$$S_o = \left\langle \frac{\cos(\cdot)}{\sqrt{\pi}}, \frac{\sin(\cdot)}{\sqrt{\pi}}, \frac{15}{\sqrt{26\pi}} \left(\frac{\cos 4(\cdot)}{15} - \frac{\cos 2(\cdot)}{3}\right)\right\rangle.$$

The functions $r_1 : E^3 \to S_o$,

$$r_2 : \left\langle \frac{1}{\sqrt{2\pi}} (\cos 2(\cdot) - \cos 4(\cdot) \right\rangle \times R^2 \to E^3$$

are given by

$$r_1(b_1, b_2, c_1) = b_1 \frac{\cos(\cdot)}{\sqrt{\pi}} + b_2 \frac{\sin(\cdot)}{\sqrt{\pi}}$$

$$+ c_1 \frac{15}{\sqrt{26\pi}} \left(\frac{\cos 4(\cdot)}{15} - \frac{\cos 2(\cdot)}{3}\right),$$

$$(b_1, b_2, c_1) \in E^3,$$

and

$$r_2\left(u_1\left(\frac{\cos 2(\cdot) - \cos 4(\cdot)}{\sqrt{2\pi}}\right), \alpha_1, \alpha_2\right) = (u_1, \alpha_1, \alpha_2).$$

We notice that $\bar{m} = \bar{\bar{m}} = 3$.

Let $(b_1, b_2, c_1) \in E^3$ and consider

$$r_1(b_1, b_2, c_1) = b_1 \frac{\cos(\cdot)}{\sqrt{\pi}} + b_2 \frac{\sin(\cdot)}{\sqrt{\pi}}$$

$$+ c_1 \frac{15}{\sqrt{26\pi}} \left(\frac{\cos 4(\cdot)}{15} - \frac{\cos 2(\cdot)}{3}\right).$$

For simplicity, we denote by

$$\xi = (b_1, b_2, c_1),$$

and $\quad v = (u_1,\ \alpha_1,\ \alpha_2).$

Then

$$T_1(\tau)\ \Gamma_1(\xi) = c_1 \frac{15}{\sqrt{26\pi}} (\cos 2(\cdot) - \cos 4(\cdot)).$$

Hence, $P_1\ T_1(\tau)\ \Gamma_1(\xi) = c_1 \dfrac{15}{\sqrt{26\pi}} (\cos 2(\cdot) - \cos 4(\cdot)).$ (5.8.21)

Also, after elementary and lengthy calculations, we get

$$P_1\ N\ \Gamma_1(\xi) = \frac{1}{4\pi}\left(\left[-\frac{279}{52\sqrt{13}} c_1^3 - \frac{9}{2\sqrt{13}} b_1^2\ c_1 - \frac{27}{2\sqrt{13}} b_2^2 c_1\right] \frac{1}{\sqrt{2\pi}} (\cos 2(\cdot) - \cos 4(\cdot))\right). \tag{5.8.22}$$

From (5.8.21) and (5.8.22), we get

$$P_1(T_1(\tau)\ \Gamma_1(\xi) - N\ \Gamma_1(\xi)) = \left[\frac{15}{\sqrt{13}} c_1 + \frac{c_1}{4\pi}\left(\frac{279}{52\sqrt{13}} c_1^2 + \frac{9}{2\sqrt{13}} b_1^2 + \frac{27}{2\sqrt{13}} b_2^2\right)\right] \frac{1}{\sqrt{2\pi}}(\cos 2(\cdot) - \cos 4(\cdot)). \tag{5.8.23}$$

We take $f_1(x) = \frac{1}{8}\left(x^{(1)}(0) - x^{(1)}(\pi)\right)^3 - x^{(1)}(2\pi) = 0$

and $f_2(x) = \frac{1}{8}\left(x(0) + x(2\pi)\right)^3 + x(\pi) = 0.$

Then, simple calculation yields that

$$f_1(\Gamma_1(\xi)) = \frac{b_2^3}{\pi\sqrt{\pi}} - \frac{b_2}{\sqrt{\pi}}, \tag{5.8.24}$$

and

$$f_2(\Gamma_1(\xi)) = \frac{1}{\pi\sqrt{\pi}}\,(b_1 - \frac{4}{\sqrt{26}}\,c_1)^3 - \frac{b_1}{\sqrt{\pi}} - \frac{4}{\sqrt{26\pi}}\,c_1. \tag{5.8.25}$$

Here, we have

$$Lx = (T_1(\tau)x,0,0) \text{ and } \bar{N}x = (\frac{x^3}{2}\,,\ f_1(x),\ f_2(x)).$$

Also,

$$\psi x = (P_1(T_1(\tau)x - Nx),\ -f_1(x),\ -f_2(x)).$$

Therefore, from (5.8.23), (5.8.24) and (5.8.25), we get

$$\psi\Gamma_1(\xi) = (\Big[\frac{15}{\sqrt{13}}\,c_1 + \frac{c_1}{4\pi}\,(\frac{279}{52\sqrt{13}}\,c_1^2 + \frac{9}{2\sqrt{13}}\,b_1^2 + \frac{27}{2\sqrt{13}}\,b_2^2)\Big]\frac{1}{\sqrt{2\pi}}\,(\cos 2(.) - \cos 4(.)),$$
$$-(\frac{b_2^3}{\pi\sqrt{\pi}} - \frac{b_2}{\sqrt{\pi}}),$$
$$-\frac{1}{\pi\sqrt{\pi}}\,(b_1 - \frac{4}{\sqrt{26}}\,c_1)^3 + \frac{b_1}{\sqrt{\pi}} + \frac{4}{\sqrt{26\pi}}\,c_1).$$

Since $\Psi = \Gamma_2\,\psi\,\Gamma_1$, we get

$$\Psi(\xi) = (\frac{15}{\sqrt{13}}\,c_1 + \frac{c_1}{4\pi}\,(\frac{279}{52\sqrt{13}}\,c_1^3 + \frac{9}{2\sqrt{13}}\,b_1^2 + \frac{27}{2\sqrt{13}}\,b_2^2),$$
$$-(\frac{b_2^3}{\pi\sqrt{\pi}} - \frac{b_2}{\sqrt{\pi}}),$$
$$-\frac{1}{\pi\sqrt{\pi}}\,(b_1 - \frac{4}{\sqrt{26}}\,c_1)^3 + \frac{b_1}{\sqrt{\pi}} + \frac{4}{\sqrt{26\pi}}\,c_1). \tag{5.8.26}$$

We clearly notice that $\xi_0 = (0,0,0)$ is a solution of $\Psi(\xi) = 0$.

We take $x_0 = 0 \in S_0$. We take the norm $|||\cdot|||$ on $H^1(J)$ and μ on $\tilde{H}^1(J)$. Consider the sets

$$U = \{ \xi \in E^3 : |\xi - \xi_0| = |\xi| \leq \varepsilon \} \tag{5.8.27}$$

and

$$V = \{ x \in S_0 : |||x-x_0|||=|||x||| \leq c,\ \mu(\dot{x}) \leq r \} \tag{5.8.28}$$

where ε, c and r are defined subsequently.

Let $\xi \in E^3$. Then simple calculation yields that

$$|||\Gamma_1(\xi)||| = \sqrt{2\pi}\,(\sup_{t \in [0,2\pi]} |(\Gamma_1(\xi))(t)|) + ||(\Gamma_1(\xi))^{(1)}||$$

$$\leq \sqrt{2}\left[|b_1| + |b_2| + \frac{6}{\sqrt{26}}|c_1|\right]$$

$$+ \frac{1}{\sqrt{\pi}}\left[\pi b_1^2+\pi b_2^2+\frac{15^2 c_1^2}{26}\left(\frac{4}{9}\pi + \frac{16}{225}\pi\right)\right]^{1/2}$$

$$\leq \sqrt{2}\left[1+1+\frac{36}{26}\right]^{1/2}\left[|b_1|^2+|b_2|^2+|c_1|^2\right]^{1/2}$$

$$+ \left(b_1^2 + b_2^2 + \frac{58}{13}c_1^2\right)^{1/2}$$

$$\leq \left(\left(\frac{88}{13}\right)^{1/2} + \left(\frac{58}{13}\right)^{1/2}\right)|\xi|$$

$$< 4.74\,|\xi|. \tag{5.8.29}$$

Also, we get

$$\mu(\Gamma_1(\xi)) = \max \left(\sup_{t \in [0,2\pi]} |\Gamma_1(\xi)(t)|, \sup_{t \in [0,2\pi]} |(\Gamma_1(\xi))^{(1)}(t)| \right)$$

$$\leq \max \left(\frac{|b_1|}{\sqrt{\pi}} + \frac{|b_2|}{\sqrt{\pi}} + \frac{6|c_1|}{\sqrt{26\pi}}, \right.$$

$$\left. \frac{|b_1|}{\sqrt{\pi}} + \frac{|b_2|}{\sqrt{\pi}} + \frac{14|c_1|}{\sqrt{26\pi}} \right)$$

$$\leq \max \left(\frac{1}{\sqrt{\pi}} (1+1+ \frac{36}{26})^{1/2} |\xi|, \frac{1}{\sqrt{\pi}} (1+1 + \frac{196}{26})^{1/2} |\xi| \right)$$

$$= (\frac{248}{26\pi})^{1/2} |\xi|$$

$$< 1.76 \ |\xi|. \qquad (5.8.30)$$

Let us take $\varepsilon_1 = \min (\frac{r}{1.76}, \frac{c}{4.74})$

$$> \min (0.56r, 0.21c).$$

Later we shall choose r and c such that

$$0.21c < 0.56r. \qquad (5.8.31)$$

Then we have

$$\varepsilon_1 > 0.21c. \qquad (5.8.32)$$

Let us take $\varepsilon = 0.21c$. (5.8.33)

For this value of ε, from (5.8.29), (5.8.30), (5.8.31) and (5.8.32), it is clear that the map Γ_1 takes the set U into V. We observe that $\Psi : U \to E^3$ has continuous partial derivatives in the interior of U.

Determination of δ : From (5.8.26), we have

$$\text{Jacobian matrix for } \Psi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{5.8.33}$$

where

$$a_{11} = \frac{9}{4\pi\sqrt{13}}\, b_1\, c_1, \quad a_{12} = \frac{27}{4\pi\sqrt{13}}\, b_2\, c_1,$$

$$a_{13} = \frac{15}{\sqrt{13}} + \frac{837}{208\pi\sqrt{13}}\, c_1^2 + \frac{1}{4\pi}\left(-\frac{9}{2\sqrt{13}}\, b_1^2 + \frac{27}{2\sqrt{13}}\, b_2^2\right),$$

$$a_{21} = 0, \quad a_{22} = -\frac{3b_2^2}{\pi\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}, \quad a_{23} = 0,$$

$$a_{31} = -\frac{1}{\pi\sqrt{\pi}}\left(3b_1^2 - \frac{24b_1c_1}{\sqrt{26}} + \frac{24}{13}\, c_1^2\right) + \frac{1}{\sqrt{\pi}},$$

$$a_{32} = 0,$$

$$a_{33} = -\frac{1}{\pi\sqrt{\pi}}\left(-\frac{96}{13\sqrt{26}}\, c_1^2 - \frac{12}{\sqrt{26}}\, b_1^2 + \frac{48}{13}\, b_1\, c_1\right) + \frac{4}{\sqrt{26\pi}}.$$

Let A_0 denote the Jacobian matrix for Ψ at $\xi_0 = (0,0,0)$. Then

$$A_0 = \begin{pmatrix} 0 & 0 & \frac{15}{\sqrt{13}} \\ 0 & \frac{1}{\sqrt{\pi}} & 0 \\ \frac{1}{\sqrt{\pi}} & 0 & \frac{4}{\sqrt{26\pi}} \end{pmatrix}. \tag{5.8.34}$$

Let A_0^{-1} denote the inverse of A_0. Calculating, we get

$$\bar{A}_o^{1} = \begin{pmatrix} -\frac{2\sqrt{2}}{15} & 0 & \sqrt{\pi} \\ 0 & \sqrt{\pi} & 0 \\ \frac{\sqrt{13}}{15} & 0 & 0 \end{pmatrix}. \tag{5.8.35}$$

Simple calculation yields that

$$\text{norm of } \bar{A}_o^{1} = \left(\frac{8}{225} + \pi + \pi + \frac{13}{225}\right)^{1/2} < 2.53. \tag{5.8.36}$$

Let the matrix $B(\xi)$ be defined as follows :

$$B(\xi) = \text{Jacobian matrix for } \psi \text{ at } \xi - \text{Jacobian matrix for } \psi \text{ at } \xi_o. \tag{5.8.37}$$

Then, making use of (5.8.33) and (5.8.34), simple and lengthy calculation yields that

$$\text{norm of } B(\xi) < 4.3\ \epsilon^2 \tag{5.8.38}$$

for all $\xi \in U$.

We make use of the following result to obtain δ.

Result : Let X and Z be Banach spaces with norms denoted by $||\cdot||_X$ and $||\cdot||_Z$ respectively. Let G be a continuous function defined for $||x||_X < \gamma_o$, with values in Z and $G(0) = 0$. Let K_o be a linear continuous transformation of X into the whole of Z, and let K_o be one-to-one. Suppose

$$||G(x_1) - G(x_2) - K_o(x_1 - x_2)||_Z \leq \delta_1 \; ||x_1 - x_2||_X \tag{5.8.39}$$

for $||x_1|| \leq \gamma_o$ and $||x_2|| \leq \gamma_o$, where δ_1 is a positive constant.

Let M be a positive number such that if $K_o(x) = z$, then $||x||_X \leq M||z||_Z$. Suppose **M** is independent of x and $M\delta_1 < 1$. Then the equation $z = G(x)$ has a solution x with $||x||_X < \gamma_o$ whenever

$$||z|| < \rho = \frac{\gamma_o(1 - M\delta_1)}{M}. \qquad (5.8.40)$$

For a proof of the above result we can refer to L.M. GRAVES; 'Some mapping theorems', Duke Math. Jour. 17 (1950) pp. 111-114.

We take $X = Y = E^3$, $K_o = A_o$, $G = \Psi$ and $\gamma_o = \varepsilon = 0.21c$. Then we get $M = 2.53$. Using mean-value theorem, we can also choose $\delta_1 = 4.3\varepsilon^2$. We shall choose c sufficient small such that $M\delta_1 < 1$. Hence, by the above result we get that the equation

$$\Psi(\xi) = v$$

has a solution ξ with $|\xi| < \varepsilon$ whenver $|v| < \rho = \frac{\gamma_o(1 - M\delta_1)}{M}$.

We can also check that the inverse function is continuous.

Calculating, we get

$$\rho = \frac{\varepsilon(1 - M\delta_1)}{M} = \frac{0.21c\ (1 - 10.879\ \varepsilon^2)}{2.53}$$

$$> 0.0819c\ (1 - 0.48069c^2).$$

We take

$$\delta = 0.0819c\ (1 - 0.48069c^2). \qquad (5.8.41)$$

We notice that c should be chosen sufficiently small such that

$$0.48069c^2 < 1. \qquad (5.8.42)$$

Determination of k_o, ℓ_1 and ℓ_2 :

We have $Nx = \frac{x^3}{2}$. Therefore,

$$|Nx-Ny| = \frac{1}{2}\,|x^3-y^3| = \frac{1}{2}\,|(x-y)(x^2+xy+y^2)|$$

$$\leq \frac{1}{2}\,|x-y|\ (|x|^2 + |x|\ |y| + |y|^2).$$

Then for $x,y \in \tilde{S}_o$, we get

$$|Nx - Ny| \leq \frac{1}{2}\, 3\bar{R}^2\ |x-y|.$$

Therefore, we have

$$k_o = \frac{3}{2}\,\bar{R}^2. \tag{5.8.43}$$

Also, let $x,y \in T(\tilde{S}_o)$. Then

$$|f_1(x)-f_1(y)| \leq \frac{1}{8}\,|(x^{(1)}(0)-x^{(1)}(\pi))^3 - (y^{(1)}(0)-y^{(1)}(\pi))^3|$$
$$+ |x^{(1)}(2\pi)-y^{(1)}(2\pi)|$$

$$\leq \frac{1}{8}\,|(x^{(1)}(0)-x^{(1)}(\pi)-y^{(1)}(0)+y^{(1)}(\pi)) \times$$
$$\Big[(x^{(1)}(0)-x^{(1)}(\pi))^2$$
$$+ (x^{(1)}(0)-x^{(1)}(\pi))(y^{(1)}(0)-y^{(1)}(\pi))$$
$$+ (y^{(1)}(0)-y^{(1)}(\pi))^2\Big]| + |x^{(1)}(2\pi)-y^{(1)}(2\pi)|$$

$$\leq \frac{1}{8}\,(|x^{(1)}(0)-y^{(1)}(0)|+|x^{(1)}(\pi)-y^{(1)}(\pi)|) \times$$
$$\Big[(|x^{(1)}(0)| + |x^{(1)}(\pi)|)^2$$
$$+ (|x^{(1)}(0)|+|x^{(1)}(\pi)|)(|y^{(1)}(0)|+|y^{(1)}(\pi)|)$$

$$+ \left(|y^{(1)}(0)| + |y^{(1)}(\pi)|\right)^2 \Big] + |x^{(1)}(2\pi) - y^{(1)}(2\pi)|$$

$$\leq \frac{1}{8} \left(2 \sup_{t \in [0,2\pi]} |x^{(1)}(t) - y^{(1)}(t)| \left[(2\bar{R})^2 + 4\bar{R}^2 + (2\bar{R})^2\right]\right)$$

$$+ \sup_{t \in [0,2\pi]} |x^{(1)}(t) - y^{(1)}(t)|$$

$$= (3\bar{R}^2 + 1) \max_{i=0,1} \sup_{t \in [0,1]} |x^{(i)}(t) - y^{(i)}(t)|.$$

Therefore, we have $\ell_1 = 3\bar{R}^2 + 1.$ (5.8.44)

Similarly, we get $\ell_2 = 3\bar{R}^2 + 1.$ (5.8.45)

Since $x_0 = \Gamma_1(\xi_0) = 0$, we clearly have

$$e = \bar{e} = 0. \tag{5.8.46}$$

To apply the theorem 5.10, first of all, our c should satisfy the following :

$$0.21c < 0.56r, \; 0.48069c^2 < 1. \tag{5.8.47}$$

By remark 5.2, the conditions of theorem 5.10 are equivalent to

$$\begin{aligned}
&\theta_1 k_0 < (5.59) \times (1.5\bar{R}^2) < 1, \; 0 < c < d, \; 0 < r < \bar{R}, \\
&c + e = c \leq (1 - 8.385\bar{R}^2) \cdot d < (1 - \theta_1 k_0) d, \\
&r + \bar{e} = r \leq (\bar{R} - 2.145\bar{R}^2 d) < \bar{R} - \bar{\theta}_1 k_0 d, \\
&(\theta_1 k_0 d + e) k_0 + (\ell_1^2 + \ell_2^2)^{1/2} (\bar{\theta}_1 k_0 d + \bar{e}) \\
&\leq (8.385\bar{R}^2 d) \times (1.5\bar{R}^2) + \sqrt{2}(3\bar{R}^2 + 1) \times (2.145\bar{R}^2 d) \\
&\leq \delta = 0.0819\, c\, (1 - 0.48069\, c^2).
\end{aligned} \tag{5.8.48}$$

One possible choice for the quantities c, r, d, $\bar{R}$ is

$$c = 0.1,\ r = 0.05,\ d = 0.2,\ \bar{R} = 0.1.$$

We can easily check that for this choice the above relations (5.8.47) and (5.8.48) are satisfied.

Hence, by remark 5.2 and theorem 5.10, the nonlinear MPBVP (5.8.1) has a solution $\hat{x}$ over the interval $[0,2\pi]$. Moreover,

$$|\hat{x}(t)| \leq 0.1 \text{ and } |(\hat{x})^{(1)}(t)| \leq 0.1.$$

APPENDIX 1

Let L be the operator defined by (1.1.3). We now show that $\dim N(L) \geq \dim N(L^*)$.

We recollect the following:

Let $G^n(J)$ be defined by

$G^n(J) = \{ y : y$ is a real-valued function on J and y together with its derivatives upto order $(n-1)$ are continuous on J except with discontinuities of first kind at each of the interior points $a_1, a_2, \ldots, a_{h-1}$ and $y^{(n)} \in S\}$.

Let $x \in H^n(J)$ and $y \in G^n(J)$. Then integration by parts yields that

$$\int_a^b (y \cdot \tau x - x \cdot \tau^* y)\,dt = B(x) \cdot B_c^*(y) + B_c(x) \cdot B^*(y). \qquad (1)$$

Here $B(x)$ = column vector $(B_1(x), B_2(x), \ldots, B_k(x))$ where $B_1, \ldots, B_k$ are defined by (1.1.2); $B^*(y)$ = column vector $(B_1^*(y), B_2^*(y), \ldots, B^*_{(h+1)n-k}(y))$ where B_j^*, $j=1,2,\ldots,(h+1)n-k$ are a set of $(h+1)n-k$ linearly independent boundary forms and each $B_j^*(y)$ contains $y^{(i)}(a)$, $y^{(i)}(a_q-0)$, $y^{(i)}(a_q+0)$, $y^{(i)}(b)$, $q=1,2,\ldots,h-1$, $i=0,1,\ldots,n-1$; '$\cdot$' denotes the usual scalar product given by

$$f \cdot g = g^* f = \sum_{i=1}^{m} g_i f_i$$

for $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$, column vectors.

B_c, B_c^* denote complementary form to B and B^*, respectively. Hence the operator L^* is defined by

$$D(L^*) = \{ y \in G^n(J) : B_j^*(y) = 0,\ j=1,2,\ldots,(h+1)n-k \},$$
$$L^* y = \tau^* y. \tag{2}$$

Obviously, B is of rank k and B^* is of rank $(h+1)n-k$. Let the matrix $(\alpha_0 : \alpha_1 : \ldots : \alpha_h)$ be defined by

$$(\alpha_0 : \alpha_1 : \ldots : \alpha_h) = \begin{pmatrix} \alpha_{010} & \alpha_{011} \cdots \alpha_{01(n-1)} \cdots & \alpha_{h10} \alpha_{h11} \cdots \alpha_{h1(n-1)} \\ \alpha_{020} & \alpha_{021} \cdots \alpha_{02(n-1)} \cdots & \alpha_{h20} \alpha_{h21} \cdots \alpha_{h2(n-1)} \\ - - - & - - - - - - - - & - - - - - - - - \\ \alpha_{0k0} & \alpha_{0k1} \cdots \alpha_{0k(n-1)} \cdots & \alpha_{hk0} \alpha_{hk1} \cdots \alpha_{hk(n-1)} \end{pmatrix} \tag{3}$$

where α_{qji}, $q=0,\ldots,h$, $j=1,2,\ldots,k$, $i=0,\ldots,n-1$ are the real constants in (1.1.2).

Then we know that rank of $(\alpha_0 : \ldots : \alpha_n)$ is k. Moreover, if $\xi =$ column vector $(x, x^{(1)}, \ldots, x^{(n-1)})$, then by (3) the boundary conditions (1.1.2) become

$$B(x) = (\alpha_0)\,\xi(a_0) + (\alpha_1)\,\xi(a_1) + \ldots + (\alpha_h)\,\xi(a_h). \tag{4}$$

Suppose B_c is a boundary form of rank $(h+1)n-k$ complimentary to B, then the matrix representation of B_c is

$$B_c = (\overset{+}{\alpha}_{0c} : \bar{\alpha}_{1c} : \overset{+}{\alpha}_{1c} : \ldots : \bar{\alpha}_{(h-1)c} : \overset{+}{\alpha}_{(h-1)c} : \bar{\alpha}_{hc})$$

where each $\overset{+}{\alpha}_{jc}$, $\bar{\alpha}_{jc}$, consists of n columns and $(h+1)n-k$ rows.

For any $x \in D(L)$, we have

$$B_c(x) = (\overset{+}{\alpha}_{0c})\xi(a_0) + \{(\bar{\alpha}_{1c}) + (\alpha^{+}_{1c})\}\xi(a_1)+\cdots+(\bar{\alpha}_{hc})\xi(a_h) \tag{5}$$

where ξ = colum vector $(x, x^{(1)}, \ldots, x^{(n-1)})$.

Let $\phi_1, \ldots, \phi_p \in N(L)$ form an orthonormal basis for $N(L)$. It will be first proved that the vectors $B_c(\phi_i)$ $(i=1,\ldots,p)$ are linear independent. Suppose they are not. Then for some constants $c_1, \ldots, c_p$, not all zero, we have

$$\sum_{i=1}^{p} c_i B_c \phi_i = 0,$$

which implies that $\sum_{i=1}^{p} B_c(c_i\phi_i) = 0$. That is

$$B_c\left(\sum_{i=1}^{p} c_i\,\phi_i\right) = 0. \tag{6}$$

However, $B\left(\sum_{i=1}^{p} c_i\,\phi_i\right) = 0,$ (7)

since $\phi_1, \ldots, \phi_p \in N(L)$. Thus, if $\tilde{\phi} = \sum_{i=1}^{p} c_i\phi_i$, and the corresponding vector ξ is $\tilde{\xi}$, then, from (6) and (7), we have

$$(\alpha_0)\tilde{\xi}(a_0) + (\alpha_1)\tilde{\xi}(a_1)+\cdots+(\alpha_h)\tilde{\xi}(a_h) = 0$$

and (8)

$$(\overset{+}{\alpha}_{0c})\tilde{\xi}(a_0) + [(\bar{\alpha}_{1c})+(\overset{+}{\alpha}_{1c})]\tilde{\xi}(a_1)+\cdots+(\bar{\alpha}_{hc})\xi(a_h)=0.$$

Since rank of

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \cdots\cdots & \alpha_h \\ \overset{+}{\alpha}_{0c} & \bar{\alpha}_{1c}+\overset{+}{\alpha}_{1c} & \cdots & \bar{\alpha}_{hc} \end{pmatrix} = (h+1)n ,$$

it follows that $\tilde{\xi}(a_o) = \tilde{\xi}(a_1) = \dots = \tilde{\xi}(a_h) = 0$. So, from $L\tilde{\phi} = 0$ and $\tilde{\xi}(a_o) = 0$, we obtain by uniqueness that $\tilde{\phi}(t)=0$ on J. This contradicts the definition of $\tilde{\phi}$ as a non-trivial combination of $\phi_1, \dots, \phi_p$. Hence $c_1=c_2=\dots=c_p = 0$.

Let $\psi_1, \psi_2, \dots, \psi_n$ be n linearly independent solutions of $\tau^* y = 0$. For $i=1,2,\dots,n$, we define

$$\begin{aligned} \psi_{ij} &= \psi_i \text{ on } (a_j, a_{j+1}),\ j=0,\dots,h-1 \\ &= 0 \text{ on } [a,b] - [a_j, a_{j+1}],\ j=0,\dots,h-1. \end{aligned} \tag{9}$$

We note that the collection $\{\psi_{ij}\}_{i=1,\ j=0}^{n\quad h-1}$ form a fundamental system of discontinuous solutions of $\tau^* y=0$. That is any discontinuous solution of $\tau^* y = 0$ with discontinuities of first kind at each of the interior points, can be expressed through this system. Moreover, the system $\{\psi_{ij}\}$ satisfies all all the properties as in the usual fundamental system. Let Ψ be the corresponding fundamental matrix.

That is

$$\Psi = \begin{pmatrix} \psi_{10} \cdots \psi_{1(h-1)} & ------ & ------ \\ \cdots\cdots & ------ & ------ \\ \psi_{10}^{(n-1)} \cdots \psi_{1(h-1)}^{(n-1)} & ------ & ------ \\ ------ & \psi_{20} \cdots\cdots \psi_{2(h-1)} & ------ \\ ------ & \cdots\cdots & ------ \\ ------ & \psi_{20}^{(n-1)} \cdots \psi_{2(h-1)}^{(n-1)} & ------ \\ ------ & ------ & ------ \\ ------ & ------ & \psi_{no} \cdots\cdots \psi_{n(h-1)} \\ ------ & ------ & \cdots\cdots \\ ------ & ------ & \psi_{no}^{(n-1)} \cdots \psi_{n(h-1)}^{(n-1)} \end{pmatrix}.$$

We clearly have

$$\int_a^b (\psi_{ij}(\tau\phi_\ell)-\phi_\ell(\tau^* \psi_{ij})=0, \ \ell=1,2,\dots,n, i=1,\dots,n, \quad j=0,\dots,h-1.$$

From the above relation, it easily follows that

$$B(\phi_\ell)\, B_c^*(\psi_{ij}) + B_c(\phi_\ell)\, B^*(\psi_{ij}) = 0 \quad \text{(see (1))}. \tag{10}$$

Since $B(\phi_\ell) = 0$, $\ell=1,2,\dots,p$, from (10) we get

$$B_c(\phi_\ell)\, B^*(\psi_{ij})=0, \ \ell=1,2,\dots,p, i=1,2,\dots,n, j=0,\dots,h-1.$$

Since $f.g. = g^* f$ for any column vectors f and g, we have

$$(B^*\Psi)^* B_c\phi_\ell = 0, \quad \ell=1,2,\dots,p.$$

Hence, the system $(B^*\Psi)^* v = 0$ has p linearly independent $(h+1)n-k$-dimensional vectors $B_c\phi_1,\dots,B_c\phi_p$ as solutions.

Therefore,

$$\text{rank of } (B^*\Psi) = \text{rank of } (B^*\Psi)^* \leq ((h+1)n-k)-p.$$

Moreover, it is true that rank of $(B^*\Psi) = ((h+1)n-k)-p$.

Also, $B^*\Psi$ is a matrix of order $((h+1)n-k) \times hn$.

We also have

$$\text{rank of } (B^*\Psi) + \text{dimension of null space of } (B^*\Psi) = \text{number of columns of } (B^*\Psi)$$

Therefore,

$$\text{dimension of null space of } (B^*\Psi) = nh-((h+1)n-k)+p = p+k-n.$$

This implies that there exist $p+k-n$ linearly independent solutions of $L^*y = 0$. But, we know that $p \geq p+k-n$ (since $k \leq n$). Let $q=p+k-n$. Hence, $\dim N(L) = p \geq \dim N(L^*) = q$. We also note that $p-q = n-k$. This proves our assertion.

APPENDIX 2

Proof of Lemma 2.1.

Let $\{x_m\}$ be a bounded sequence in $H^n(J)$. Then there exists $c_o > 0$ such that

$$\sqrt{b-a}\left(\sum_{i=0}^{n-1} \sup_{t \in J} |x_m^{(i)}(t)|\right) + ||x_m^{(n)}|| \leq c_o \tag{1}$$

for all m. Hence, for all m we have

$$\sup_{t \in J} |x_m^{(i)}(t)| \leq c_o/\sqrt{b-a}, \; i=0,1,\ldots,n-1 \tag{2}$$

and

$$||x_m^{(n)}|| \leq c_o. \tag{3}$$

We know that, for $1 < p < \infty$, the space $AC^p(J)$ is defined as the space of all those functions defined everywhere on J for which

$$\sum_{\Delta \in \pi} \frac{|\Delta f|^p}{(\Delta x)^{p-1}}$$

remains bounded over all partitions π of J where Δ is the usual difference operator.

We recollect the following result from functional analysis:

Statement: $f \in AC^2(J)$ if and only if $f^{(1)} \in S$ and $||f^{(1)}|| = ||f||_{AC^2}$ where

$$||f||_{AC^2} = (\sup_{\pi}(\sum_{\Delta \in \pi} \left|\frac{\Delta f}{\Delta x}\right|^2))^{1/2} . \tag{4}$$

Making use of the above result, for all m we have

$$|x_m^{(n-1)}(t_1)-x_m^{(n-1)}(t_2)| \leq ||x_m^{(n)}|| \; |t_1-t_2|^{1/2}, \quad t_1, t_2 \in J. \tag{5}$$

Then, by (3), we get

$$|x_m^{(n-1)}(t_1)-x_m^{(n-1)}(t_2)| \leq c_o|t_1-t_2|^{1/2} \quad \text{for all } m. \tag{6}$$

Therefore, from (2) and (6), we have that the sequence $\{x_m^{(n-1)}\}$ is uniformly bounded and equicontinuous on J. Hence, by Ascoli's theorem, the sequence $\{x_m^{(n-1)}\}$ has a uniformly convergent subsequence. Moreover, making use of (2) and the meanvalue theorem, repeated application of Ascoli's theorem to the subsequences yields that there exist a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ and a function $x \in \mathbf{C}^{n-1}(J)$ such that the sequence $\{x_{m_k}^{(i)}\}$ converges to uniformly to $x^{(i)}$, $i=0,\ldots,n-1$. We note that $x \in H^{n-1}(J)$. Obviously, the sequence $\{x_{m_k}\}$ converges to x in the topology of $H^{n-1}(J)$. This completes the proof of Lemma 2.1.

APPENDIX 3

We shall find an $\varepsilon > 0$ such that $\Gamma_1(U) \subseteq V$ where V and Γ_1 are defined by (2.3.3) and (2.7.5), respectively and U is the set corresponding to ε defined by (2.7.8).

We take the operator τ defined by (1.1.1). We know that $\Gamma_1(\tau) = x \in S_0$. Let $x \in S_0$ be any arbitrary element. From Theorem 0.1, we have

$$\sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t \in J} |x^{(i)}(t)|\Big) + ||x^{(n)}|| \le \hat{b}(||x||+||\tau x||) \tag{1}$$

where $\hat{b} > 0$ is a number independent of x.

On the other hand, since S_0 is finite dimensional and the operator L is linear, for all $x \in S_0$ we have

$$||\tau x|| \le \hat{\hat{b}}\, ||x||, \tag{2}$$

where $\hat{\hat{b}} > 0$ is some number independent of x.

Then, from (1) and (2), we get

$$\sqrt{b-a}\Big(\sum_{i=0}^{n-1} \sup_{t \in J} |x^{(i)}(t)|\Big) + ||x^{(n)}|| \le \hat{b}(1+\hat{\hat{b}})\, ||x|| \tag{3}$$

for all $x \in S_0$, which readily implies that

$$|||x||| \le \tilde{b}\, ||x||, \quad \mu(x) \le \tilde{\tilde{b}}\, ||x|| \tag{4}$$

where $\tilde{b} = \hat{b}(1+\hat{\hat{b}})$ and $\tilde{\tilde{b}} = \dfrac{\hat{b}(1+\hat{\hat{b}})}{\sqrt{b-a}}$.

We now take $\varepsilon = \min\left(\frac{c}{\tilde{b}} , \frac{r}{\tilde{\tilde{b}}} \right)$. (5)

Corresponding to this ε , we consider the set U defined by (2.7.8).

Let $\xi \in U$. Then $r_1(\xi) - r_1(\xi_o) = x-x_o$ and $||x-x_o|| = |\xi-\xi_o| \leq \varepsilon$. Moreover, from (4) and (5), we get

$$|||r_1(\xi)-r_1(\xi_o)||| \leq \tilde{b}\; ||r_1(\xi)-r_1(\xi_o)|| = \tilde{b}\; |\xi-\xi_o|$$
$$\leq \tilde{b}\; \varepsilon \leq c,$$

and

$$\mu(r_1(\xi)- r_1(\xi_o)) \leq \tilde{\tilde{b}}\; |\xi-\xi_o| \leq \tilde{\tilde{b}}\; \varepsilon \leq r.$$

Thus we found an $\varepsilon > 0$ such that $r_1(U) \subsetneq V$.

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